Subject Code: MA201BS Lecture no: 1 Date: 4/6/21

Subject Name: Mathematics-II

Topic Name: Double Integrals & Evaluation of Double Integrals.

Name of the faculty: V. Sankar Rao UNIT-III Sem-I/II.

Introduction:- In lower classes we have seen how to use the method of integration to caliculate the area of a bounded region. we shall now how to use the method of double integration to calucuIate the area. these idea can be extended to mulltiple integrals. ie integrals of functions of more than two variables.

A double integral is known as Multiple integrals is an extension of a Definite integral of a function of single variable to a function of two (or) three variables. The multiple integral is usually called as improper when the region of integration extends to infinity (or) when the integral becomes infinite at a point inside the region (or) on the boundary of the region. These are useful ine evaluating area, volume mass, centroid and moment of inertia in plane and solid regions.

Double Integrals :- Consider a region R in the xy-plane bounded by one (or) more curves. Let $f(x,y)$ be a function defined at all points of R. Let the region R be devided into small sub regions each of Area $\delta R_1, \delta R_2 \cdots \delta R_n$ which

which are pairwise non-overlapping.

Let (x_i, y_i) be an arbitary point within the region δR_i.

consider the sum $f(x_1,y_1)\delta R_1 + f(x_2,y_2)\delta R_2 + \cdots + f(x_n,y_n)\delta R_n$

If this sum tends to a finite limit as $n\to\infty$

such that $\max(\delta R_i)\longrightarrow 0$ irrespective of the choice of (x_i, y_i), the limit is called the "double integral" of $f(x,y)$ over the region R and is denoted by the symbol

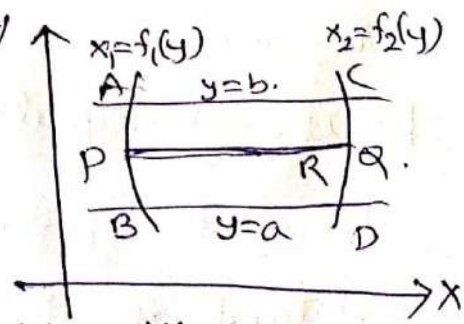
$$\boxed{\iint_R f(x,y)\,dR \quad (\text{or}) \quad \iint_R f(x,y)\,dx\,dy.}$$

Evaluation of Double Integrals:

→ Evaluation of a double integrals depends upon the nature of the curves bounding the Region R. Let region R bounded by the curves $x=x_1$, $x=x_2$ and $y=y_1$ and $y=y_2$.

Case (i) :- when x_1 and x_2 are functions of y where as y_1 and y_2 are constants.

Let the equations of the curves AB and CD be $x=x_1=f_1(y)$ and $x=x_2=f_2(y)$ and let y_1 and y_2 be Constants.



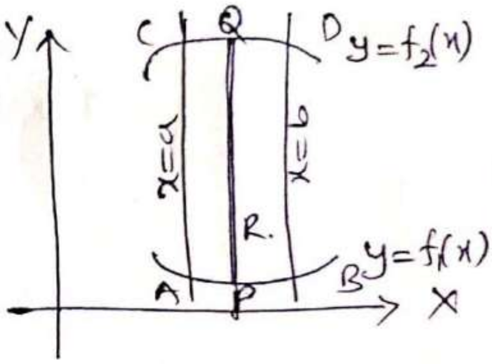
→ Take a horizontal strip PQ of width δy. Then the double integration of a function $f(x,y)$ over the region R is evaluated first w.r.t 'x' (treating y as a Constant). The resulting expression which will be a function y is integrated w.r.t y between the limits $y=a$ and $y=b$.

Hence $$\iint_R f(x,y)\,dx\,dy = \boxed{\int_{y=a}^{y=b} \boxed{\int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\,dx}\,dy}$$

the integration is carried from the ~~into~~ inner to the outer rectangle.

Case (ii):- when y_1 and y_2 are functions x where as x_1 and x_2 are constants.

→ let AB & CD be the curves, $y = y_1 = f_1(x)$ and, $y = y_2 = f_2(x)$ and $x = a$, $x = b$ are constants.



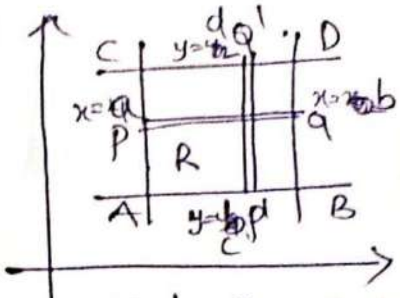
consider a vertical strip PQ of width δx. In this case the double integration of $f(x,y)$ over the region R is first evaluated by integrating $f(x,y)$ w.r.t 'y' (treating x as a constant) between the limits $y = \cancel{f_1(x)}\ f_1(x)$ to $y = f_2(x)$. The resulting expression which will be function of x is then integrated w.r.t 'x' between the limits $x = a$ and $x = b$.

$$\therefore \iint_R f(x,y)\,dx\,dy = \boxed{\int_{x=a}^{x=b} \boxed{\int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\,dy}\,dx}$$

The integration is carried from the inner to the outer rectangle.

Case (iii) :- when x_1, x_2, y_1, y_2 are all constants.

In this case, the region of integration ABCD, ie 'R' is a rectangle since x_1, x_2, y_1, y_2 are all constants.



The order of integration is immaterial provided the limits of integration are changed accordingly.

Thus,
$$\iint_R f(x,y)\,dx\,dy = \boxed{\int_{y=c}^{y=d} \boxed{\int_{x=a}^{x=b} f(x,y)\,dx}\,dy.}$$

(OR)

$$\iint_R f(x,y)\,dx\,dy = \boxed{\int_{x=a}^{x=b} \boxed{\int_{y=c}^{y=d} f(x,y)\,dy}\,dx}.$$

Note (1) :- Suppose if the limits, for one variable is a function of the other and constants for other variable, then first we have to integrate the variable w.r.t to variable for which limits are function of the other, then the variable with constant limits.

Note (2) :- while evaluating a multiple integral w.r.t one variable, the other variables are treated as constansts.

Subject code : MA201BS. Lecture no : 2 Date : 5/6/21

Subject Name : Mathematics - II

Topic Name : Examples of Double Integration (Cartesian form).

Name of the faculty : V. Sankar Rao Unit - III Sem - I/II.

Introduction :- In this section we will discuss on problems of double integration (cartesian form). Suppose if the limits for one variable is a function of the other and constants for other variable, then first we have to integrate the variable with respect to variable for which limits are function of the other, then the variable with constant limits. Also while evaluating a double integral with respect to one variable, the other variables are treated as constants.

Problems :-

(1) Evaluate $\int_{y=0}^{2}\int_{x=0}^{3} xy \, dx \, dy$.

Sol :- given $\int_{y=0}^{2}\int_{x=0}^{3} xy \, dx \, dy = \int_{y=0}^{2}\left[\int_{x=0}^{3} xy \, dx\right] dy$.

First we have to integrate the inner integral w.r.t x keeping y as a constant.

$$\therefore = \int_0^2 \left[y \cdot \left(\frac{x^2}{2}\right)_0^3 dy \right]$$

$$= \int_0^2 \left(\frac{9y}{2}\right) dy = \frac{9}{2}\int_0^2 y \, dy$$

$$= \frac{9}{2} \cdot \left[\frac{y^2}{2}\right]_0^2 = \frac{9}{2}\left[\frac{4}{2}\right] = 9.$$

2) Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx$.

<u>Sol</u>:- Here all the four limits are constants. So the double integral can be evaluated in either way. ie we first integrate w.r.t 'x' and then w.r.t 'y' (or) we first integrate w.r.t 'y' and then w.r.t 'x'.

$$\therefore \text{ given integral} = \int_0^3 \left[\int_1^2 (xy + x^2y + xy^2)\,dy\right] dx.$$

$$= \int_0^3 \left[x\left(\frac{y^2}{2}\right)_1^2 + x^2\left(\frac{y^2}{2}\right)_1^2 + x\left(\frac{y^3}{3}\right)_1^2\right) dx$$

$$= \int_0^3 \left(x\left(\frac{4}{2}-\frac{1}{2}\right) + x^2\left(\frac{4}{2}-\frac{1}{2}\right) + x\left(\frac{8}{3}-\frac{1}{3}\right)\right) dx$$

$$= \int_0^3 x\left(\frac{3}{2}\right) + x^2\left(\frac{3}{2}\right) + x\left(\frac{7}{3}\right) dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_0^3 + \frac{3}{2}\left(\frac{x^3}{3}\right)_0^3 + \frac{7}{3}\left(\frac{x^2}{2}\right)_0^3$$

$$= \frac{3}{2}\left(\frac{9}{2}-0\right) + \frac{3}{2}\left(\frac{27}{3}-0\right) + \frac{7}{3}\left(\frac{9}{2}-0\right)$$

$$= \frac{3}{2}\times\frac{9}{2} + \frac{\cancel{3}}{2}\times\frac{27}{\cancel{3}} + \frac{7}{\cancel{3}}\times\frac{\cancel{9}^{3}}{2}$$

$$= \frac{27}{4} + \frac{27}{2} + \frac{21}{2}$$

$$= \frac{27+54+42}{4} = \frac{123}{4}.$$

imp

3). Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$.

<u>Sol</u>:- Here the limits of the interior integration are functions of x. Hence we must understand that these are limits of 'y'.

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→ so integrate first w.r.t 'y' and next w.r.t 'x'.

$$\int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}}(x^2+y^2)\,dy\right]dx = \int_0^1\left(x^2y+\frac{y^3}{3}\right)_x^{\sqrt{x}}dx$$

$$= \int_0^1\left(x^2\sqrt{x}+\frac{(\sqrt{x})^3}{3}\right)-\left(x^2\cdot x+\frac{x^3}{3}\right)dx$$

$$= \int_0^1\left(x^{5/2}+\frac{x^{3/2}}{3}-4\frac{x^3}{3}\right)dx$$

$$= \left(\frac{x^{7/2}}{7/2}+\frac{x^{5/2}}{3\cdot\frac{5}{2}}-\frac{4}{3}\cdot\frac{x^4}{4}\right)_0^1$$

$$= \frac{2}{7}+\frac{2}{15}-\frac{1}{3}$$

$$= \frac{30+14-35}{105} = \frac{\cancel{9}^{3}}{\cancel{105}_{35}} = \frac{3}{35}.$$

4). Evaluate $\int_0^4\int_{y^2/4}^{y}\frac{y}{x^2+y^2}\,dx\,dy$.

Sol:- Here the limits of the interior integration are functions of y. so we must understand that these are limits of 'x'. So we integrate first w.r.t x and next w.r.t 'y'.

$$= \int_0^4\left[\int_{y^2/4}^{y}\frac{y}{x^2+y^2}\,dx\right]dy.$$

$$= \int_0^4 y\left(\int_{y^2/4}^{y}\left(\frac{1}{x^2+y^2}\right)dx\right)dy. \quad \left[\because\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}(x/a)\right]$$

$$= \int_0^4 y\left[\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)_{y^2/4}^{y}\right]dy.$$

$$= \int_0^4 \left(\tan^{-1}(1) - \tan^{-1}\left(\frac{y}{4}\right)\right) dy.$$

$$= \int_0^4 \left(\frac{\pi}{4} - \tan^{-1}(y/4)\right) dy$$

$$= \frac{\pi}{4}\int_0^4 dy - \int_0^4 \tan^{-1}\left(\frac{y}{4}\right) dy \quad \text{(Integration by parts).}$$

$$= \frac{\pi}{4}\left[y\right]_0^4 - \left[\left(\tan^{-1}\left(\frac{y}{4}\right)\cdot y\right)_0^4 - \int_0^4 y \cdot \frac{1}{1+\frac{y^2}{16}} \cdot \frac{1}{4}\, dy\right)$$

$$= \frac{\pi}{4}(4-0) - \left[4\cdot\tan^{-1}(1) - 4\cdot\int_0^4 \frac{y}{y^2+16}\, dy\right]$$

$$= \pi - \left[\cancel{4}\cdot\frac{\pi}{\cancel{4}} - 2\left(\log(y^2+16)\right)_0^4\right] \cdot \quad \left[\int \frac{f'(x)}{f(x)} dx = \log f(x)\right].$$

$$= \cancel{\pi} - \cancel{\pi} + 2\left(\log(32) - \log(16)\right]$$

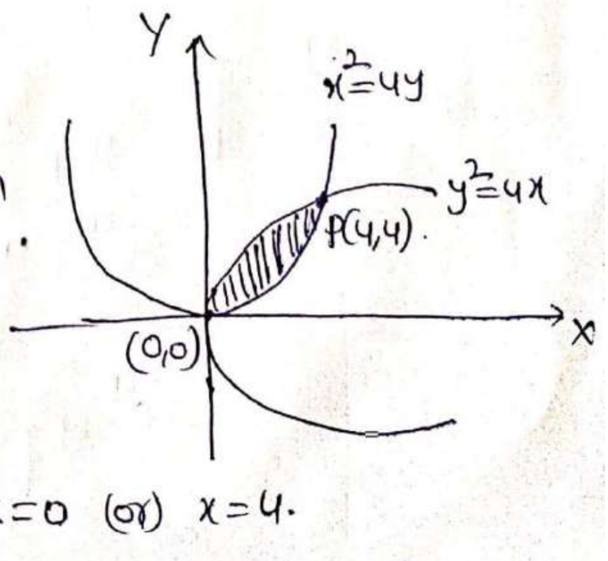
$$= 2\log\left(\frac{\cancel{32}^{\,2}}{\cancel{16}}\right) = 2\log 2$$

imp

5). Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Sol:- Given parabolas $y^2 = 4x$ — ① &

$x^2 = 4y$ — ②



To find their points of intersection

Solve eq ① and eq ②

$$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 4^3 x$$

$$\Rightarrow x(x^3 - 4^3) = 0$$

$$x^3 = 4^3 \Rightarrow x = 0 \text{ (or) } x = 4.$$

Thus the two parabolas intersect at the points $(0,0)$ and $P(4,4)$.

→ The shaded area between the parabolas eq ① & eq ② is the region of integration.

$$\iint_R y\,dx\,dy = \int_{x=0}^{4}\int_{y=x^2/4}^{2\sqrt{x}} y\,dx\,dy = \int_0^4\left[\int_{y=x^2/4}^{2\sqrt{x}} y\,dy\right]dx$$

$$= \int_0^4 \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[4\cdot\left(\frac{x^2}{2}\right)_0^4 - \frac{1}{16}\left(\frac{x^5}{5}\right)_0^4\right)$$

$$= \frac{1}{2}\left(4\left(\frac{16}{2}-0\right) - \frac{1}{16}\left(\frac{4^5}{5}-0\right)\right]$$

$$= \frac{1}{2}\left(32 - \frac{64}{5}\right)$$

$$= \frac{48}{5}.$$

imp

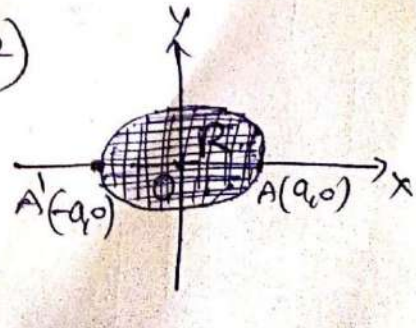
6). Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol:- given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

ie $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{1}{a^2}(a^2-x^2)$

$\therefore y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$\therefore y = \pm\frac{b}{a}\sqrt{a^2-x^2}$.



Hence the region of integration R can be expressed as

$$-a \le x \le a, \quad -\frac{b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}.$$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_{x=-a}^{a} \int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy.$$

$$= 2\int_{-a}^{a} \left[\int_{0}^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dy\right] dx$$

$$= 2\int_{-a}^{a} \left(x^2 y + \frac{y^3}{3}\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\cdot\int_{-a}^{a} \left(x^2\cdot\frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right)dx$$

$$= 4\int_0^a \left(\frac{b}{a}x^2\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right)dx$$

put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$.

at $x=0 \Rightarrow \theta = 0$

$x = a \Rightarrow \theta = \pi/2$.

$$= 4\int_0^{\pi/2} \left(\frac{b}{a}a^2\sin^2\theta\cdot a\cos\theta + \frac{b^3}{3a^3}a^3\cos^3\theta\right)\cdot a\cos\theta\,d\theta$$

$$= 4\int_0^{\pi/2} \left(a^3 b\sin^2\theta\cdot\cos^2\theta + \frac{ab^3}{3}\cos^4\theta\right)d\theta$$

$$= 4\left[a^3 b\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{ab^3}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right].$$

$$= \frac{4\pi}{16}(a^3b + ab^3)$$

$$= \frac{\pi ab(a^2+b^2)}{4}.$$

7) Evaluate $\iint_R xy \, dx\,dy$ where R is the region bounded by the line $x+2y=2$ lying in the first quadrant.

Sol :- The Region R is bounded by the lines $y=0$, $y=\frac{1}{2}(2-x)$, $x=0$ and $x=2$.

Hence $$\iint_R xy\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{\frac{1}{2}(2-x)} xy\,dy\,dx$$

$$= \int_{x=0}^{2} x\cdot\left(\frac{y^2}{2}\right)_0^{\frac{1}{2}(2-x)} dx$$

$$= \frac{1}{2}\cdot\int_0^2 \frac{x}{4}(2-x)^2\,dx$$

$$= \frac{1}{8}\int_0^2 (4x-4x^2+x^3)\,dx$$

$$= \frac{1}{8}\left[4\cdot\frac{x^2}{2} - 4\cdot\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2$$

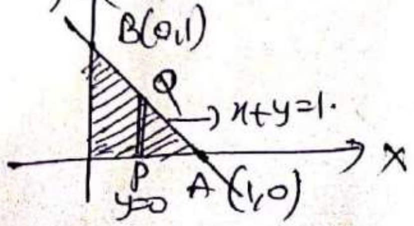
$$= \frac{1}{8}\left(8-\frac{32}{3}+4\right)$$

$$= \frac{1}{8}\left[12-\frac{32}{3}\right] = \frac{1}{24}(4) = \frac{1}{6}.$$

8) Evaluate $\iint (x^2+y^2)\,dx\,dy$ in the positive quadrant for which $x+y\le 1$.

Sol :- $$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)\,dx\,dy$$

$$= \int_{x=0}^{x=1}\left(x^2y+\frac{y^3}{3}\right)_0^{1-x} dx.$$



$$= \int_{x=0}^{1} \left(x^2 - x^3 + \frac{1}{3}(1-x)^3\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4\right]_{x=0}^{x=1}$$

$$= \frac{1}{3} - \frac{1}{4} - 0 + \frac{1}{12}$$

$$= \frac{1}{6}.$$

Practice problems

1). Evaluate $\iint_R xy(x+y)\,dx\,dy$ over the region R bounded by $y = x^2$ and $y = x$.

2). Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$.

3). Evaluate $\iint_R (x+y)\,dx\,dy$, over the region in the positive quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

4). Evaluate $\iint_R x^2\,dx\,dy$ over the region bounded by hyperbola $xy = 4$, $y = 0$, $x = 1$, $x = 4$.

Subject code : MA201BS Lecture no : 3 Date : 7/6/21

Subject Name : Mathematics - II

Topic Name : Double Integrals in polar co-ordinates.

Name of the faculty : V. Sankar Rao Unit - III Sem - I/II

Introduction :- To evaluate over the region bounded by the lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$, $r = r_2$ we first integrate w.r.t 'r' between limits $r = r_1$ and $r = r_2$ keeping θ fixed. The resulting expression is integrated w.r.t 'θ' from θ_1 to θ_2. In this integral r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

→ Suppose $f(x,y)$ is expressed in polar co-ordinates as

$$f(r\cos\theta, r\sin\theta) = F(r,\theta).$$

Suppose we want $\iint_R f(x,y)\,dR$.

When we write the integrand interms (r,θ) the above integral becomes $\iint_R F(r,\theta)\,dR$, where R is described interms of (r,θ) and dR is the area of the element in polar co-ordinates. It is equal to $r\delta r\delta\theta = r\,dr\,d\theta$.

$\therefore$ The integral becomes $\iint_R F(r,\theta)\,r\,dr\,d\theta$.

Problems :- (1) Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

Sol :- given $\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right]d\theta$

$$= \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi} \left(\frac{a^2 \sin^2\theta}{2} \right) d\theta \qquad (\because \sin^2\theta = 1 - \cos 2\theta).$$

$$= \frac{a^2}{2} \int_0^{\pi} \frac{1-\cos 2\theta}{2} d\theta = \frac{a^2}{4} \int_0^{\pi} (1-\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{4} \left[\int_0^{\pi} 1\, d\theta - \int_0^{\pi} \cos 2\theta\, d\theta \right]$$

$$= \frac{a^2}{4} \left[(\theta)_0^{\pi} - \left(\frac{\sin 2\theta}{2} \right)_0^{\pi} \right]$$

$$= \frac{a^2}{4} \left[\pi - \left(\frac{\sin 2\pi}{2} - 0 \right) \right]$$

$$= \frac{a^2 \pi}{4}.$$

2). Evaluate $\int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a} r^2\, dr\, d\theta.$

<u>Sol</u>:- given $\int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a} r^2\, dr\, d\theta = \int_{\theta=0}^{\pi/2} \left[\int_{r_2 a(1-\cos\theta)}^{a} r^2\, dr \right] d\theta$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_{a(1-\cos\theta)}^{a} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{a^3}{3} - \frac{a^3(1-\cos\theta)^3}{3} \right] d\theta.$$

$$= \frac{a^3}{3} \left(\int_0^{\pi/2} \left(1 - (1-\cos\theta)^3\right) d\theta \right.$$

$$= \frac{a^3}{3} \left[\int_0^{\pi/2} 1 - (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\, d\theta \right.$$

2)

$$= \frac{a^3}{3}\left(\int_0^{\pi/2} (3\cos\theta - 3\cos^2\theta + \cos^3\theta)\,d\theta\right.$$

$$= \frac{a^3}{3}\left[3\int_0^{\pi/2}\cos\theta\,d\theta - 3\int_0^{\pi/2}\cos^2\theta\,d\theta + \int_0^{\pi/2}\cos^3\theta\,d\theta\right]$$

$$= \frac{a^3}{3}\left[3.(\sin\theta)_0^{\pi/2} - 3\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2}{3\cdot 1}\right].$$

$$= \frac{a^3}{3}\left[3(1) - \frac{3\pi}{4} + \frac{2}{3}\right] = \frac{a^3}{3}(44 - 9\pi).$$

3) Evaluate $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\,d\theta\,dr$

Sol:- given $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2}\, r\,d\theta\,dr = \int_0^{\infty} e^{-r^2} r.\left[\int_0^{\pi/2} d\theta\right] dr$

$$= \int_0^{\infty} r.e^{-r^2}(\theta)_0^{\pi/2}\,dr$$

$$= \frac{\pi}{2}\int_0^{\infty} r.e^{-r^2}\,dr$$

$$= -\frac{\pi}{4}\int_0^{\infty}(-2r).e^{-r^2}\,dr$$

$$= -\frac{\pi}{4}\left(e^{-r^2}\right)_0^{\infty}$$

$$= -\frac{\pi}{4}(0-1)$$

$$= \frac{\pi}{4}.$$

practice questions

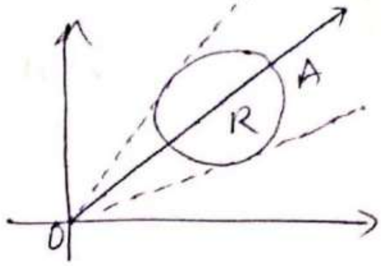
1). Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta.$

2). Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}.$

Working Rule to find the polar limits of integration

Consider the integral $\iint_R f(r,\theta)\,dr\,d\theta$ over a region R where the limits of integration are not specified.

<u>Step(1)</u> :- Sketch the region of integration R and label the bounding curves.



<u>Step(2)</u> :- Imagine a radius vector OA through the region in the direction of increasing r. Mark the values of r interms of θ, where the radius vector enters and emerges from the region.

$\therefore$ The r limits of integration are varies from $r = f_1(\theta)$ to $r = f_2(\theta)$.

<u>Step(3)</u> :- Find the smallest and largest values of θ which include the region R. This gives the limits of integration for θ.

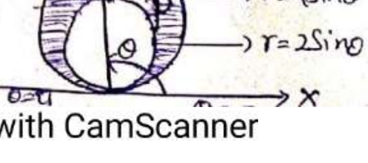
$\therefore$ The θ- limits of integration are varies from $\theta = \theta_1$ to $\theta = \theta_2$.

Hence $$\boxed{\iint_R f(r,\theta)\,dr\,d\theta = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\,dr\,d\theta}$$

<u>Problems</u> :- (1) ~~The region~~ Evaluate $\iint r^3\,dr\,d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

<u>Sol</u> :- The region of integration R is shown shaded.

Here 'r' varies from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region 'θ' varies from 0 to π.



$$\therefore \iint_R r^3\, dr\, d\theta = \int_0^{\pi} \int_{r=2\sin\theta}^{r=4\sin\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\pi} \left[\int_{r=2\sin\theta}^{r=4\sin\theta} r^3\, dr \right] d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4} \right]_{r=2\sin\theta}^{r=4\sin\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} (256 \sin^4\theta - 16 \sin^4\theta)\, d\theta$$

$$= \frac{1}{4} \cdot \int_0^{\pi} (240 \sin^4\theta)\, d\theta$$

$$= \frac{\cancel{240}^{60}}{\cancel{4}} \int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60 \cdot 2 \int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= \cancel{120}^{\cancel{30}\,15} \times \frac{3 \times 1}{\cancel{4} \times \cancel{2}} \cdot \frac{\pi}{2}$$

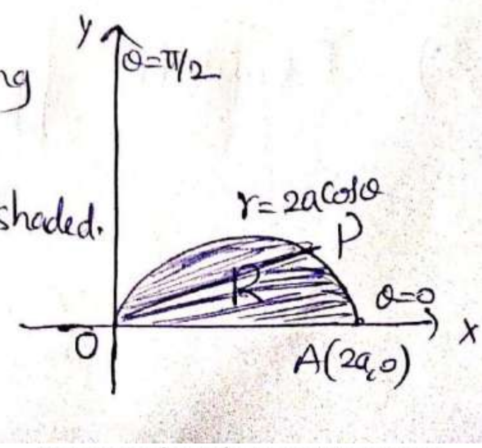
$$= \frac{45\pi}{2}.$$

2). S.T $\iint_R r^2 \sin\theta\, dr\, d\theta = \frac{2a^2}{3}$, where R is the semi-circle $r = 2a\cos\theta$ above the intial line.

Sol: The semi-circle $r = 2a\cos\theta$ passing through the pole.

The Region of integration R is shown shaded.

Here 'r' varies from 0 to $2a\cos\theta$ while θ varies from 0 to $\pi/2$.



$$\therefore \iint_R r^2 \sin\theta \, dr\, d\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \left[\int_{r=0}^{2a\cos\theta} r^2 \, dr \right] d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3} \right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin\theta \left(8a^3 \cos^3\theta - 0 \right) d\theta$$

$$= \frac{8a^3}{3} \int_0^{\pi/2} \sin\theta . \cos^3\theta \, d\theta$$

put $\cos\theta = t$
$-\sin\theta d\theta = dt$
at $\theta = 0 \Rightarrow t = 1$
$\theta = \pi/2 \Rightarrow t = 0$

$$= \frac{8a^3}{3} \left(\int_1^0 t^3 (-dt) \right)$$

$$= \frac{8a^3}{3} \left[\int_0^1 t^3 dt \right]$$

$$= \frac{8a^3}{3} \left[\frac{t^4}{4} \right]_0^1 = \frac{\cancel{8}^{2} a^3}{3} \times \frac{1}{\cancel{4}} = \frac{2a^3}{3}.$$

practice questions:

1). Evaluate $\iint_R r \sin\theta \, dr\, d\theta$ ~~over~~ over the cardioid $r = a(1-\cos\theta)$ above the intial line.

2). Evaluate $\iint \frac{r \, dr \, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of the leminscate $r^2 = a^2 \cos 2\theta$.

Subject code: MA201BS Lecture no: 4 Date: 8/6/21

Subject Name: Mathematics-II.

Topic Name: Change of Variables in double integral.

Name of the faculty: V. Sankar Rao Unit-III Sem-I/II

Introduction:- The evaluation of a double (or) triple integral with its present form may not be simple to evaluate. By coice of an appropriate co-ordinate system, a given integral can be transformed into a simpler integral involving the new variables.

Transformation of co-ordinates:

Let $x = f(u,v)$ and $y = g(u,v)$ be the relations between the old variables (x,y) with the new variables (u,v) of the new co-ordinate system.

Then $$\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv \longrightarrow ①$$

where $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

which is called the "Jacobian" of the co-ordinate transformation.

Change of Variables from Cartesian to polar co-ordinates:

In this case, we have $u=r$, $v=\theta$ and $x = r\cos\theta$, $y = r\sin\theta$.

and $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

Hence eq ① becomes

$$\iint_R F(x,y)\,dx\,dy = \iint_R F(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

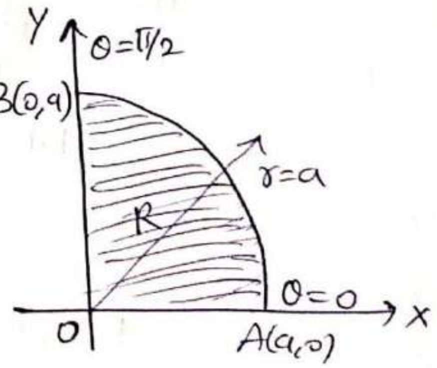
This corrsponds to $\iint F(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r,\theta)\,r\,dr\,d\theta$

Similarly, from polar to cartesian co-ordinates can be described.

<u>problems</u>:-

(1) Evaluate double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$ by changing into polar co-ordinates.

<u>Sol</u>:- given $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$ ——— ①



The region R of integration is specified by inequalities.

$$0 \le x \le \sqrt{a^2-y^2}\,,\quad 0 \le y \le a.$$

ie R is region bounded by the circle. $x^2+y^2=a^2$ in the first Qudrant.

we reduce ① into the polar form by putting $x = r\cos\theta$, $y = r\sin\theta$. and replace $dx\,dy$ by $r\,dr\,d\theta$ in the cartesian integral.

$$\therefore \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^2 \cdot r\, dr d\theta = \int_{\theta=0}^{\pi/2}\left(\frac{r^4}{4}\right)_0^a d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} d\theta$$

$$= \frac{a^4}{4}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{4}\left[\frac{\pi}{2}\right] = \frac{\pi a^4}{8}.$$

2). Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing into polar co-ordinates.

Sol:- The region of integration is given by $y=0$, $y=\sqrt{2x-x^2}$, $x=0$, $x=2$.

ie $y=0$, $x^2+y^2=2x$, $x=0$ and $x=2$

Let $x=r\cos\theta$, $y=r\sin\theta$

Then $dx\,dy = r\,dr\,d\theta$.

Now $y=0 \Rightarrow r\sin\theta=0$

$\Rightarrow \theta=0$

and $x=0 \Rightarrow r\cos\theta=0$

$\Rightarrow \theta=\pi/2$ $(r\neq 0)$.

and $y=\sqrt{2x-x^2} \Rightarrow x^2+y^2=2x \Rightarrow r^2=2(r\cos\theta)$

$\therefore r=2\cos\theta$.

Hence in polar co-ordinates the given region is bounded by the curves $r=0$, $r=2\cos\theta$, $\theta=0$ and $\theta=\pi/2$.

$$\therefore \int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx = \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2 (r\,dr\,d\theta)$$

$$= \int_0^{\pi/2} \left[\int_0^{2a\cos\theta} r^3 \, dr \right] d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2\cos\theta} d\theta$$

$$= \frac{2^4}{4} \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= \cancel{4}^{2} \, \frac{3}{\cancel{2}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\pi}{2} = \frac{3\pi}{2}.$$

imp

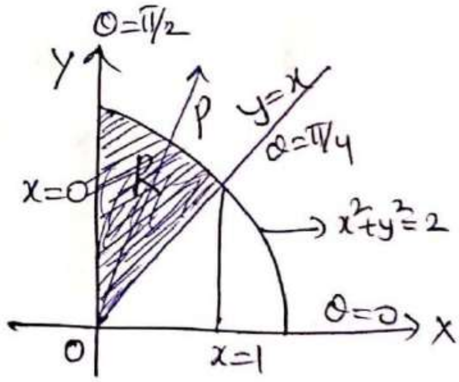
3). By changing into polar co-ordinates, evaluate

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2} \, dy \, dx.$$

Sol:- The region of integration is given by

$y = x$, $y = \sqrt{2-x^2}$, $x = 0$ and $x = 1$

ie The region is bounded by the lines $y=x$ and the circle $x^2+y^2=2$ in the first quardrant. between the lines $x=0$ and $x=1$.



To change the given double integration into polar co-ordinates put $x = r\cos\theta$, $y = r\sin\theta$ then $dx\,dy = r\,dr\,d\theta$

and $\frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{1}{r}\cos\theta$.

To find the polar limits, imagine a radius vector from O (ie $r=0$) which emerges ~~to~~ the region at P where $r = \sqrt{2}$ such radii vectors can be drawn in ~~the~~ between the lines $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$.

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Hence $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\, dy\, dx = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{\sqrt{2}} \frac{1}{r}\cos\theta \cdot r\, dr\, d\theta$

$$= \int_{\theta=\pi/4}^{\pi/2} \cos\theta\, d\theta \int_0^{\sqrt{2}} dr$$

$$= \int_{\pi/4}^{\pi/2} \cos\theta\, (r)_0^{\sqrt{2}}\, d\theta$$

$$= \sqrt{2} \int_{\pi/4}^{\pi/2} \cos\theta\, d\theta$$

$$= \sqrt{2}\, (\sin\theta)_{\pi/4}^{\pi/2} = \sqrt{2}\left(\sin\pi/2 - \sin\pi/4\right)$$

$$= \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right).$$

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{x^2+y^2}\, dy\, dx = \sqrt{2} - 1.$$

Practice questions:

1). Evaluate the following integral by transforming into polar co-ordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} y \cdot \sqrt{x^2+y^2}\, dx\, dy$.

2). Evaluate $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\, dy$ over the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using the transformations $x = au$ and $y = bv$.

Subject code : MA201BS   Lecture no: 5   Date: 9/6/21

Subject Name: Mathematics-II

Topic Name : Change of order of Integration.

Name of the faculty: V. Sankar Rao   Unit-III   Sem- I/II

Introduction :- The order of integration is responsible for the description of region and accordingly the limits of integration.

→ Here the change of order of integration implies that the change of limits of integration.

→ If the region of integration consists of a vertical strip and slide along x-axis then in the changed order a horizontal strip and slide along y-axis are to be considered and vice-versa.

→ Some times we may have to split the region of integration and express the given integral as sum of the integrals over these sub-regions.

→ Some times as commented above, the evaluation gets simplified due to change of order of integration. Always it is better to draw a rough sketch of region of the integration.

Working Procedure to change the order of Integration.

Step ①:- First identify the variables for the limits.

step ②:- Draw a rough sketch of the region of integration

Step ③:- If we are evaluating the integral with respect to 'y' first, then take a vertical strip ie a strip parallel to y-axis. Otherwise, take a horizontal strip ie a strip parallel to x-axis.

Step ④:- Now rotate the strip by an angle of $90°$ in the anti-clock wise direction and identify the starting and ending points of the strip, which will be the lower and upper limits of that variable.

Step ⑤:- Identify the limits for other variables for the region of consideration.

Step ⑥:- Evaluate the double integral with new order of integration.

problems:- (1) Change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$.

Sol:- In the given integral for a fixed x,

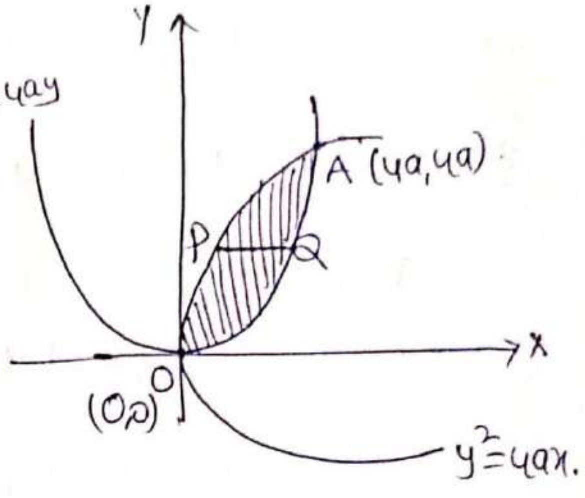
y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$

x varies from 0 to $4a$

Let the curves $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$

ie $x^2 = 4ay$ and $y^2 = 4ax$

The region of integration is the shaded portion in the figure.



The given integral is

$$\int_{0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx.$$

points of intersecting curves are $O(0,0)$ & $A(4a,4a)$.

By the change of order of integration,

we must fix y first, For a fixed y,

x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$. and then

y varies from 0 to $4a$.

Hence the integral is equal to

$$\int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a} \left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx \right] dy$$

$$= \int_{y=0}^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$\int_{y=0}^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= 2\sqrt{a}\cdot\int_{y=0}^{4a} y^{1/2}\,dy - \frac{1}{4a}\int_{y=0}^{4a} y^2\,dy$$

$$= 2\sqrt{a}\cdot\left[\frac{y^{3/2}}{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left[(4a)^{3/2}-0\right] - \frac{1}{4a}\left(\frac{(4a)^3}{3}-0\right).$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\left((2^2)^{3/2}\cdot a^{3/2}\right) - \frac{1}{\cancel{4a}}\cdot\frac{(4a)^{3/2}}{3}$$

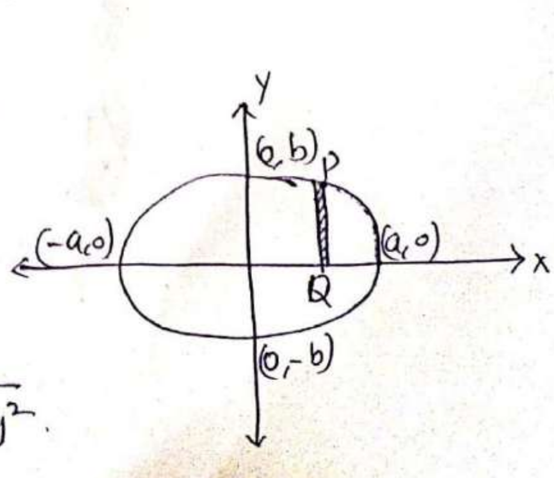
$$= 2\cdot a^{1/2}\cdot\frac{2}{3}\cdot 8\cdot a^{3/2} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} \Rightarrow \frac{32a^2-16a^2}{3}$$

$$\boxed{\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \frac{16a^2}{3}}.$$

2). Change the order of integration and evaluate

$$\int_{y=0}^{b}\int_{x=0}^{\frac{a\sqrt{b^2-y^2}}{b}} xy\,dx\,dy.$$



Sol:- Here the integration is first w.r.t x, then w.r.t y.

The limits of integration are

$x=0$ and $x = \frac{a}{b}\sqrt{b^2-y^2}$.

$y=0$ and $y=b$.

$\therefore \quad x = 0$ and $x = \frac{a}{b}\sqrt{b^2-y^2}$

squaring on bothsides

$$x^2 = \frac{a^2}{b^2}(b^2-y^2)$$

$$b^2x^2 = a^2(b^2-y^2).$$

divide by a^2b^2 on b.s.

$$\Rightarrow \quad \frac{\cancel{b^2}x^2}{a^2\cancel{b^2}} = \frac{\cancel{a^2b^2}}{\cancel{a^2b^2}} - \frac{\cancel{a^2}y^2}{\cancel{a^2}b^2}$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ (which is an ellipse equation).

The points of intersections of Co-ordinate axes are

$(-a,0)$ $(0,a)$ and $(0,-b)$ $(0,b)$.

By change of order of integration we must fix first 'y'.

$\therefore$ y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$ and

x varies from 0 to a.

Hence the integral is

$$\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a x.\left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\right]dx.$$

$$= \int_0^a x.\left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}} dx = \int_0^a \frac{x}{2}\left(\frac{b^2}{a^2}(a^2-x^2)\right)dx.$$

$$= \frac{\cancel{a^2}b^2}{2\cancel{a^2}}\int_0^a x\,dx - \frac{b^2}{2a^2}\int_0^a x^3\,dx.$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a - \frac{b^2}{2a^2}\left[\frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2}-0\right] - \frac{b^2}{2a^2}\left[\frac{a^4}{4}-0\right] = \frac{b^2}{2}\cdot\frac{a^2}{2} - \frac{b^2}{2\cancel{a^2}}\cdot\frac{\cancel{a^4}a^2}{4}$$

$$= \boxed{\frac{b^2a^2}{8}}.$$

Subject code : MA201BS Lecture no: 6 (1) Date: 17/6/21

Subject Name : Mathematics-II

Topic Name : Change of order of Integration - problems.

Name of the faculty: V. Sankar Rao Unit-III Sem- I/II

<u>Introduction</u>:- In this section we will discuss the problems related to change of order of integration. we know that change of order of integration implies that change the limits of integration.

<u>problems</u>:-

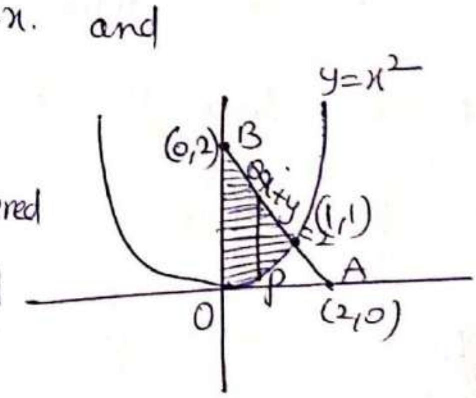
imp (1). Change the order of integration and hence evaluate the double integral $\int_0^1 \int_{x^2}^{2-x} xy \, dy dx$.

<u>Sol</u>:- Here the given ~~given~~ integral $y : x^2 \longrightarrow 2-x$ and

$x : 0 \longrightarrow 1.$

ie $y = x^2$ to $y = 2-x$. and

$x = 0$ to $x = 1$.



Here vertical strip PQ is considered ie y limits from the strip and x limits from the slide.

we shall draw a curves $y = x^2$ and $y = 2-x$.

The line $y = 2-x$ passes through $(0,2), (2,0)$

The points of intersection of the curves $y = x^2$ and $y = x-2$ are obtained by solving these two equations,

$\because$ $y = x^2$ and $y = 2-x$.

equating the values of y, we get

$$x^2 = 2-x$$

$$x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$\Rightarrow \quad x=-2, \; x=1.$$

$\therefore$ $x=-2$ gives $y=4$.

$x=1$ gives $y=1$.

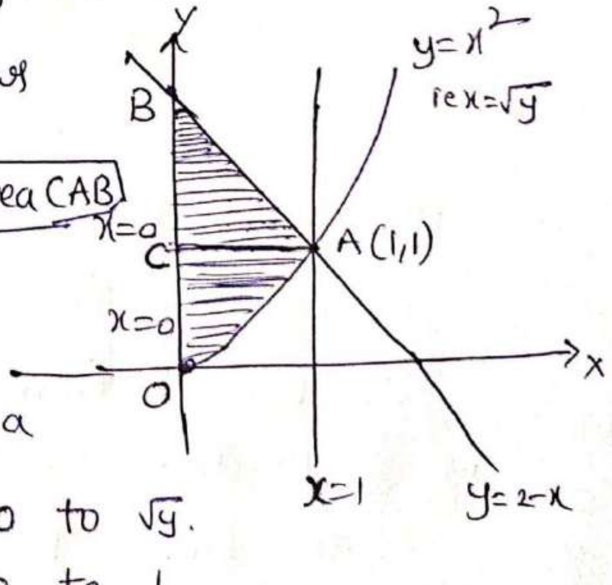
Hence the points of intersection of the curves are $(-2,4)$ $(1,1)$.

The shaded region in the figure is the region of integration.

Suppose we change the order of integration.

In the change of order of integration we have to take horizontal strips since during sliding one edge of the strip remains on $x=0$ but the other edge of the strip does not remain on a single curve.

$\therefore$ we take the region as follows



$\boxed{\text{Area of OAB} = \text{Aera OAC} + \text{Area CAB}}$

we shall fix y first.

For the region with in OACO for a fixed y, x varies from 0 to $\sqrt{y}$.

~~Then~~ Then y varies from 0 to 1.

For the region with in CABC, for a fixed y.

x varies from 0 to $2-y$

and then y varies from 1 to 2.

Hence $\int_0^1 \int_{x^2}^{2-x} xy \, dy\, dx = \iint_{OACO} xy \, dx\, dy + \iint_{CABC} xy\, dx\, dy$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y\left[\int_0^{\sqrt{y}} x\, dx\right] dy + \int_{y=1}^{2} y\left[\int_0^{2-y} x\, dx\right] dy.$$

$$= \int_{y=0}^{1} y\cdot\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_{y=1}^{2} y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \int_{y=0}^{1} y\cdot\left(\frac{(\sqrt{y})^2}{2}\right) dy + \int_1^2 y\cdot\frac{(2-y)^2}{2}\, dy.$$

$$= \int_{y=0}^{1} y\cdot\left(\frac{y}{2}\right) dy + \int_1^2 y\cdot\frac{(4+y^2-4y)}{2}\, dy.$$

$$= \int_0^1 \frac{y^2}{2}\, dy + \frac{1}{2}\int_1^2 (4y + y^3 - 4y^2)\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[4\cdot\left[\frac{y^2}{2}\right]_1^2 + \left(\frac{y^4}{4}\right)_1^2 - 4\cdot\left(\frac{y^3}{3}\right)_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}\right] + \frac{1}{2}\left[4\cdot\left(\frac{4}{2}-\frac{1}{2}\right) + \cancel{\left(\frac{2}{4}\right) - 4\left(\frac{1}{3}\right)} \left(\frac{2^4}{4}-\frac{1}{4}\right) - 4\left(\frac{2^3}{3}-\frac{1}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{12}{2} + \frac{15}{4} - \frac{28}{3}\right)$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{6}{1} + \frac{15}{4} - \frac{28}{3}\right) = \frac{1}{6} + \frac{1}{2}\left(+\frac{5}{12}\right)$$

$\int_0^1 \int_x^{2-x} xy\, dy\, dx. = \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}$

2). By changing the order of integration, evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx.$$

<u>Sol</u> :- The area of integration lies between $y=0$ and $y=\sqrt{1-x^2}$

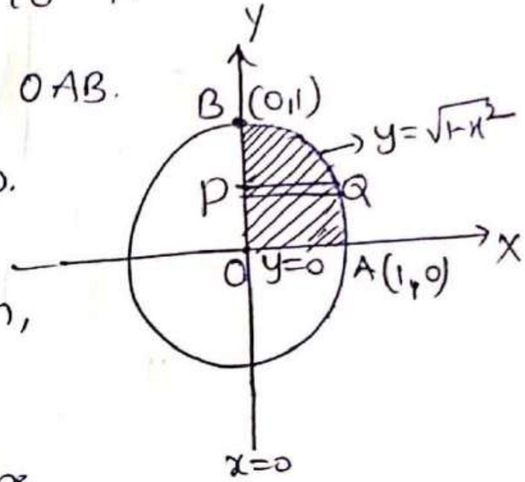
ie $y^2 = 1-x^2$

$\therefore$ $x^2+y^2=1$. which represents a circle.

The limits of x are 0 to 1.

Hence the region of integration is OAB.

and is divided into vertical strip.



For changing the order of integration, we shall divide the region of integration into horizontal strips.

The points of intersections are obtained by solving

$y=0$ and $y=\sqrt{1-x^2}$

Then $(0,1)$ $(1,0)$ are points of intesections.

The new limits of integration become

x varies from $x=0$ to $x=\sqrt{1-y^2}$

and y varies from $y=0$ to $y=1$.

$$\therefore \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dy \, dx = \int_{y=0}^{1} \left[\int_{x=0}^{\sqrt{1-y^2}} y^2 \, dy \right] dx$$

$$= \cancel{\int_0^1 \left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx}.$$

(3)

$$\therefore \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2\, dy\, dx$$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2\, dy\, dx$$

$$= \int_{y=0}^{1} y^2 \left[\int_{x=0}^{\sqrt{1-y^2}} dx \right] dy.$$

$$= \int_0^1 y^2 \left[x \right]_0^{\sqrt{1-y^2}} dy.$$

$$\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 dy dx = \int_0^1 y^2 \left(\sqrt{1-y^2}\right) dy.$$

put $y = \sin\theta$

$dy = \cos\theta\, d\theta$

Then $y=0 \Rightarrow \theta = 0$

$y = 1 \Rightarrow \theta = \pi/2.$

$$= \int_0^{\pi/2} \sin^2\theta \left(\sqrt{1-\sin^2\theta}\right) \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta . \cos\theta . \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta\, d\theta$$

Comparing the above integral to $\left[\text{w.k.t} \int_0^{\pi/2} \sin^{2m-1}\theta . \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m, \right.$

$\therefore$

$2m-1 = 2,$	$2n-1 = 2$
$2m = 3$	$2n = 3$
$m = 3/2$	$n = 3/2.$

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$$\int_{y=0}^{1}\int_{0}^{\sqrt{1-y^2}} y^2\,dy\,dx = \int_{0}^{\pi/2} \sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\,\beta\left(\frac{3}{2}, \frac{3}{2}\right).$$

$$\boxed{\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}$$

$$= \frac{1}{2}\cdot \frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$$= \frac{1}{2}\cdot\frac{\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{6}{2}\right)}$$

$$= \frac{1}{2}\cdot\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{\Gamma(3)}$$

$$= \frac{1}{2}\cdot\frac{\frac{\pi}{4}}{2!} = \frac{1}{2}\cdot\frac{\pi}{4}\times\frac{1}{2} = \frac{\pi}{16}.$$

Practice problems:

imp 1). Change the order of integration and evaluate

$$\int_{0}^{a}\int_{\frac{x^2}{a}}^{2a-x} xy^2\,dy\,dx.$$

2). change the order of integration and evaluate

$$\int_{0}^{a}\int_{\frac{x}{a}}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy.$$

3). Change the order of integration in the integral

$$\int_{0}^{a/2}\int_{x^2/a}^{x-\left(\frac{x^2}{a}\right)} f(x,y)\,dy\,dx.$$

Subject code : MA201BS Lecture No: 7 Date: 19/6/21

Subject Name: Mathematics -II.

Topic Name : Triple Integrals.

Name of the faculty: V. Sankar Rao UNIT-III, Sem- I/II

Introduction :- In this section we will discuss the concept of triple integral. In the earlier sections, we have discussed the concept of a double integral over a region R in xy-plane. now this can be extended further to define a triple integral.

Def :- Let $f(x,y,z)$ be a function which is defined at all points in a finite region V in space. Let $\delta x \delta y \delta z$ be an elementary volume of V enclosing of the point (x,y,z). Thus the triple summation

$$\underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{Lt} \sum \sum \sum f(x,y,z)\, \delta x\, \delta y\, \delta z$$

If it exists is written as $\iiint f(x,y,z)\, dx\, dy\, dz$ which is called the triple integral of $f(x,y,z)$ over the region V.

If the region V is bounded by the surfaces

$x = x_1,\ x = x_2;\ y = y_1,\ y = y_2;\ z = z_1,\ z = z_2$

then,
$$\boxed{\iiint_V f(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z)\, dx\, dy\, dz} \rightarrow (1)$$

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Evaluation of Triple Integrals :

To evaluate the triple integral given by eq ①, we proceed as follows.

Case (i) :- If $x_1, x_2 ; y_1, y_2 ; z_1, z_2$ are all constants, then the order of integration is immaterial provided the limits of integration are changed accordingly.

$$\boxed{\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz = \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dz\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dy\,dx}$$

Case (ii) :- If z_1, z_2 are functions of x and y.

y_1, y_2 are functions of x.

and x_1, x_2 are constants.

Then the integration must be performed first w.r.t z, second w.r.t y and finally w.r.t x.

Hence

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_{x=a}^{x=b}\left[\int_{y=f_1(x)}^{y=f_2(x)}\left[\int_{z=g_1(x,y)}^{z=g_2(x,y)} f(x,y,z)\,dz\right]dy\right]dz$$

where the integration is carried out from the inner most rectangle to the outer most rectangle.

Note :- If $f(x,y,z)=1$ then the integral

$$\boxed{\iiint_V dx\,dy\,dz = \iiint_V dv.}$$

Problems :-

(1) Evaluate $\int_0^1 \int_1^2 \int_2^3 xyz \, dx \, dy \, dz$

Sol:- given integral limits are all constants so that order of integration is immaterial.

$$\therefore \int_0^1 \int_1^2 \int_2^3 xyz \, dx \, dy \, dz = \int_0^1 \int_1^2 \left[\int_2^3 xyz \, dx \right] dy \, dz$$

$$= \int_0^1 \int_1^2 yz \left[\frac{x^2}{2} \right]_2^3 dy \, dz$$

$$= \int_0^1 \int_1^2 \left(\frac{3^2}{2} - \frac{4}{2} \right) yz \, dy \, dz$$

$$= \frac{5}{2} \int_0^1 \left[\int_1^2 yz \, dy \right] dz$$

$$= \frac{5}{2} \int_0^1 z \left[\frac{y^2}{2} \right]_1^2 dz$$

$$= \frac{5}{2} \int_0^1 z \left(\frac{4}{2} - \frac{1}{2} \right) dz$$

$$= \frac{5}{2} \cdot \frac{3}{2} \cdot \left[\frac{z^2}{2} \right]_0^1$$

$$= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}$$

$$\boxed{\int_0^1 \int_1^2 \int_2^3 xyz \, dx \, dy \, dz = \frac{15}{8}}$$

2). Evaluate $\int_0^1 \int_y^1 \int_0^{1-x} x \, dz \, dx \, dy$.

Sol:- Here the given integral, order of integration is first w.r.t z, second w.r.t x finally w.r.t y.

$$= \int_0^1 \int_y^1 \left[x \int_0^{1-x} 1 \, dz \right] dy \, dx$$

$$= \int_0^1 \int_y^1 x \cdot [z]_0^{1-x} \, dy \, dx$$

$$= \int_0^1 \left[\int_y^1 x(1-x) \, dx \right] dy.$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_y^1 dy$$

$$= \int_0^1 \left[\left(\frac{1}{2} - \frac{y^2}{2} \right) - \left(\frac{1}{3} - \frac{y^3}{3} \right) \right) dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{y^2}{2} - \frac{y^3}{3} \right) dy$$

$$= \int_0^1 \left(\frac{1}{6} + \frac{y^3}{3} - \frac{y^2}{2} \right) dy.$$

$$= \frac{1}{6} [y]_0^1 + \frac{1}{3} \left[\frac{y^4}{4} \right]_0^1 - \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{6} [1] + \frac{1}{3} \left(\frac{1}{4} \right) - \frac{1}{2} \left(\frac{1}{3} \right)$$

$$\int_0^1 \int_y^1 \int_0^{1-x} x \, dz \, dx \, dy = \cancel{\frac{1}{6}} + \frac{1}{12} - \cancel{\frac{1}{6}} = \frac{1}{12}.$$

3). Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy.$

Sol:- The order of integration first w.r.t z, second w.r.t 'x' finally w.r.t y (∵ from given limits)

$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy = \int_1^e \int_1^{\log y} \left[\int_1^{e^x} \log z \, dz \right] dx \, dy$$

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$$\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy = \int_1^e \int_1^{\log y} \left[z\cdot\log z - z\right]_1^{e^x} dx \, dy$$

$$\left[\because \int \log x \, dx = x(\log x - 1)\right].$$

$$= \int_1^e \left[\int_1^{\log y} \left[x\cdot e^x - e^x + 1\right] dx\right] dy$$

$$= \int_1^e \left[x\cdot e^x - e^x - e^x + x\right]_1^{\log y} dy$$

$$= \int_1^e \left(x\cdot e^x - 2e^x + x\right)_1^{\log y} dy$$

$$= \int_1^e \left(y\cdot\log y - 2y + \log y\right) - \left(e - 2e + 1\right) dy$$

$$= \int_1^e \left(y \log y + \log y - 2y + e - 1\right) dy.$$

$$= \left[\left(\frac{y^2}{2} + y\right)\log y - \left(\frac{y^2}{4} + y\right) - y^2 + (e-1)\cdot y\right]_1^e$$

$$= \left(\left(\frac{e^2}{2} + e\right)\log e - \left(\frac{e^2}{4} + e\right) - e^2 + (e-1)\cdot e\right) - \left(\left(\frac{1}{2}+1\right)\log 1 - \left(\frac{1}{4}+1\right) - 1 + (e-1)1\right].$$

$$= \frac{e^2}{2} + \cancel{e} - \frac{e^2}{4} - \cancel{e} - \cancel{e^2} + \cancel{e^2} - e + \frac{5}{4} + 1 - e + 1 \qquad \left(\begin{matrix}\log e = 1 \\ \log 1 = 0\end{matrix}\right).$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4}$$

$$\therefore \int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy = \frac{e^2}{4} - 2e + \frac{13}{4}.$$

imp

4) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz$.

Sol:- The limits of inner integral are functions of x & y.

So these are limits of z.

So first integrate w.r.t z, second integrate w.r.t y.

and finally w.r.t x.

$$\therefore \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz = \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} dz \right] dy\, dx.$$

$$= \int_0^1 \int_0^{1-x} [z]_0^{1-x-y}\, dy\, dx$$

$$= \int_0^1 \left[\int_0^{1-x} (1-x-y)\, dy \right] dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx.$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1-x)^2\, dx$$

$$= \frac{1}{2} \left[\frac{(1-x)^3}{-3} \right]_0^1 = -\frac{1}{6} \left((1-x)^3 \right)_0^1$$

$$= -\frac{1}{6} \left(0 - (1-0) \right)$$

$$= -\frac{1}{6}(-1) = \frac{1}{6}.$$

— x —.

Lecture no: 8 (1) Date: 21/6/21

Subject code : MA201BS

Subject Name: Mathematics-II

Topic Name: Triple Integral - problems.

Name of the faculty: V. Sankar Rao Unit -III Sem- I/II

<u>Introduction</u> :- In this section we will discuss the problems related to triple integrals.

<u>Problems</u>:-

(1). Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

<u>Sol</u>:- The limits of inner integral are functions of x and y. middle integral is function of x.

So first integrate w.r.t z, second integrate w.r.t y and finally w.r.t x.

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx. = \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^x.e^y.e^z\, dz\, dy\, dx$$

$$= \int_0^{\log 2}\int_0^{x} e^x e^y \left[e^z\right]_0^{x+\log y} dy\, dx$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^{x+\log y} - e^0\right) \cancel{dy}\ \cancel{dx}.e^x e^y\, dy\, dx.$$

$$= \int_0^{\log 2}\int_0^{x} \left(e^x \cdot e^{\log y} - 1\right) \cancel{dy}\ \cancel{dx}.e^x e^y\, dy\, dx$$

$$= \int_0^{\log 2}\int_0^{x} e^x.e^y\left(e^x y - 1\right) dy\, dx.$$

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$$= \int_0^{\log 2} e^x \int_0^x e^y (e^x y - 1)\, dy\, dx$$

$$= \int_0^{\log 2} e^x \left[(ye^x - 1)e^y - \int e^x e^y\, dy \right]_0^x dx$$

$$= \int_0^{\log 2} e^x \left[(ye^x - 1)e^y - e^{x+y} \right]_0^x dx$$

$$= \int_0^{\log 2} e^x \left[(xe^x - 1)e^x - e^{2x} + 1 + e^x \right] dx$$

$$= \int_0^{\log 2} e^x \left(xe^{2x} - \cancel{e^x} - e^{2x} + 1 + \cancel{e^x} \right) dx$$

$$= \int_0^{\log 2} \left(xe^{3x} - e^{3x} + e^x \right) dx$$

$$= \left[x \cdot \frac{e^{3x}}{3} - \int 1 \cdot \frac{e^{3x}}{3}\, dx - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2}$$

$$= \left[\frac{x}{3} \cdot e^{3x} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x \right]_0^{\log 2}$$

$$= \left(\frac{\log 2}{3} e^{3\log 2} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{3\log 2} \right) - \left(0 - \frac{e^0}{9} - \frac{e^0}{3} + e^0 \right).$$

$$= \left(\frac{8}{3}\log 2 - \frac{8}{9} - \frac{8}{3} + 2 \right) - \left(\frac{-1}{9} - \frac{1}{3} + 1 \right)$$

$$\boxed{\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+y}\, dx\, dy\, dz = \frac{8}{3}\log 2 - \frac{19}{9}.}$$

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2) Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ where V is the region of space bounded by $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$.

Sol :- $$\iiint_V (xy+yz+zx)\,dx\,dy\,dz = \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}(xy+yz+zx)\,dx\,dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left[y\cdot\frac{x^2}{2}+yz(x)+z\cdot\frac{x^2}{2}\right]_0^1 dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left[y\left(\frac{1}{2}\right)+yz+\frac{z}{2}\right]dy\,dz$$

$$= \int_{z=0}^{3}\frac{1}{2}\left(\frac{y^2}{2}\right)_0^2+z\left[\frac{y^2}{2}\right]_0^2+\frac{z}{2}\left(y\right)\Big]_0^2 dz$$

$$= \int_0^3\left(\frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{2}}{2}\right)+z\left(\frac{\cancel{4}^{2}}{\cancel{2}}\right)+\frac{z}{\cancel{2}}(\cancel{2})\right)dz$$

$$= \int_0^3(1+2z+z)\,dz$$

$$= \left[z+2\cdot\left(\frac{z^2}{2}\right)+\left(\frac{z^2}{2}\right)\right]_0^3$$

$$= 3+\cancel{2}\left(\frac{9}{\cancel{2}}\right)+\frac{9}{2}$$

$$= 12+\frac{9}{2}$$

$$= \frac{24+9}{2}$$

$$\boxed{\iiint_V (xy+yz+zx)\,dx\,dy\,dz = \frac{33}{2}}$$

3). Evaluate $\iiint_V dx\,dy\,dz$ where V is the finite region of space formed by the planes $x=0, y=0, z=0$, and $2x+3y+4z=12$.

Sol:- given equation of is $2x+3y+4z=12$

$z=0$ varies from $z=\frac{1}{4}(12-2x-3y)$

y varies from $y=0$ to $y=\frac{1}{3}(12-2x)$.

x varies from $x=0$ to $x=6$.

$$\iiint_V dx\,dy\,dz = \int_{x=0}^{6}\int_{y=0}^{\frac{1}{3}(12-2x)}\int_{z=0}^{\frac{1}{4}(12-2x-3y)} dx\,dy\,dz$$

$$= \int_0^6\int_0^{\frac{1}{3}(12-2x)} [z]_0^{\frac{1}{4}(12-2x-3y)}\,dy\,dz$$

$$= \int_0^6\int_0^{\frac{1}{3}(12-2x)} \frac{1}{4}(12-2x-3y)\,dy\,dz$$

$$= \frac{1}{4}\int_0^6\int_0^{\frac{1}{3}(12-2x)} (12-2x-3y)\,dy\,dz$$

$$= \frac{1}{4}\int_0^6 \left[12(y)-2x(y)-3\left(\frac{y^2}{2}\right)\right]_0^{\frac{1}{3}(12-2x)} dx$$

$$= \frac{1}{4}\int_0^6 \left[\cancel{12}^{4}\left(\frac{1}{\cancel{3}}(12-2x)\right)-2x\left(\frac{1}{3}(12-2x)\right)-\frac{3}{2}\left(\frac{1}{3}(12-2x)\right)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \left(48-8x-\frac{\cancel{24}^{8}x}{\cancel{3}}+\frac{4x^2}{3}-\frac{\cancel{3}}{2}\left(\frac{1}{\cancel{9}3}(12-2x)^2\right)\right)dx$$

$$\iiint_V dx\,dy\,dz = \frac{1}{4}\int_0^6 48-8x-8x+\frac{4x^2}{3}-\frac{1}{6}\left(144+4x^2-48x\right)dx$$

$$=\frac{1}{4}\int_0^6 \left(48-16x+\frac{4x^2}{3}-\frac{\cancel{144}^{24}}{\cancel{6}}-\frac{\cancel{4}^{2}x^2}{\cancel{6}_{3}}+\frac{\cancel{48}^{8}x}{\cancel{6}}\right)dx$$

$$=\frac{1}{4}\int_0^6\left(48-8x+\frac{4}{3}x^2-\frac{2}{3}x^2-24\right)dx$$

$$=\frac{1}{4}\int_0^6\left(24+\frac{2}{3}x^2-8x\right)$$

$$=\frac{1}{4}\left(24[x]+\frac{2}{3}\left(\frac{x^3}{3}\right)-8\left(\frac{x^2}{2}\right)\right]_0^6$$

$$=\frac{1}{4}\left[24(6)+\frac{2}{3}\left(\frac{6^3}{3}\right)-\cancel{8}^{4}\left(\frac{6^2}{\cancel{2}}\right)\right]$$

$$=\frac{1}{4}\left(\cancel{144}+\frac{2}{\cancel{3}}\times\frac{\cancel{72}^{24}\cancel{216}}{\cancel{3}}-\cancel{144}\right]$$

$$=\frac{\cancel{48}^{12}}{\cancel{4}}=12.$$

4). Evaluate $\iiint xyz\,dz\,dx\,dy$ over the positive quadrant of the sphere $x^2+y^2+z^2=a^2$.

Sol:- Given sphere is $x^2+y^2+z^2=a^2 \Rightarrow z=\sqrt{a^2-x^2-y^2}$

The projection of the sphere on the xy-plane is the circle $x^2+y^2=a^2$.

So, this circle is covered as y varies from '0' to $\sqrt{a^2-x^2}$.

x varies from 0 to a.

$$\iiint_V xyz\,dxdydz = \int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$\iiint_V xyz\,dx\,dy\,dz = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[yx \int_{z=0}^{\sqrt{a^2-x^2-y^2}} z\,dz \right] dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} yx\,dy\,dx.$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} yx \left(\frac{\left(\sqrt{(a^2-x^2-y^2)}\right)^2}{2}\right) dy\,dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} yx\,(a^2-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2} \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\,dy\right] dx$$

$$= \frac{1}{2} \int_0^a \left[a^2x \cdot \left(\frac{y^2}{2}\right) - x^3\left(\frac{y^2}{2}\right) - x\left(\frac{y^4}{4}\right)\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a a^2x \left(\frac{\left(\sqrt{a^2-x^2}\right)^2}{2}\right) - x^3\left(\frac{\left(\sqrt{a^2-x^2}\right)^2}{2}\right) - x\left(\frac{\left(\sqrt{a^2-x^2}\right)^4}{4}\right) dx$$

$$= \frac{1}{2} \int_0^a \left(a^2x\,\frac{(a^2-x^2)}{2} - x^3\left(\frac{a^2-x^2}{2}\right) - x \cdot \frac{(a^2-x^2)(a^2-x^2)}{4}\right) dx$$

$$= \frac{1}{2} \int_0^a \left(\frac{a^4x - a^2x^3}{2} - \left(\frac{a^2x^3 - x^5}{2}\right) - \frac{x(a^4 - 2a^2x^2 + x^4)}{4}\right) dx$$

$$= \frac{1}{2} \int_0^a \left(\frac{2a^4x - 2a^2x^3 - 2a^2x^3 + 2x^5 - a^4x + 2a^2x^3 - x^5}{4}\right) dx$$

$$= \frac{1}{2} \times \frac{1}{4} \int_0^a (a^4x - 4a^2x^3 + x^5 + 2a^2x^3)\,dx$$

$$\iiint_V xyz\,dx\,dy\,dz = \frac{1}{8}\int_0^a \left(a^4x - 4a^2x^3 + 2a^2x^3 + x^5\right)dx$$

$$= \frac{1}{8}\left[a^4.\left[\frac{x^2}{2}\right]_0^a - 4a^2\left(\frac{x^4}{4}\right)_0^a + 2a^2\left(\frac{x^4}{4}\right)_0^a + \left(\frac{x^6}{6}\right)_0^a\right]$$

$$= \frac{1}{8}\left[a^4.\left(\frac{a^2}{2}\right) - \cancel{4}a^2.\frac{a^4}{\cancel{4}} + \cancel{2}a^2\left(\frac{a^4}{\cancel{4}2}\right) + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\frac{a^6}{2} - \frac{a^6}{1} + \frac{a^6}{2} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\frac{\cancel{3a^6} - \cancel{6a^6} + \cancel{3a^6} + a^6}{6}\right]$$

$$\boxed{\iiint_V xyz\,dx\,dy\,dz = \frac{a^6}{48}}$$

Practices Questions:

1). Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$

2) Evaluate $\displaystyle\int_0^1\int_0^{1-z}\int_0^{1-y-z} xyz\,dx\,dy\,dz.$

3). Evaluate $\displaystyle\iiint_V \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ taken over the volume bounded by the planes $x=0, y=0, z=0$ and the plane $x+y+z=1$.

4). Evaluate $\displaystyle\int_0^1\int_0^2\int_1^2 x^2yz\,dx\,dy\,dz.$

Subject code: MA201BS   Lecture no: 9   Date: 22/5/21

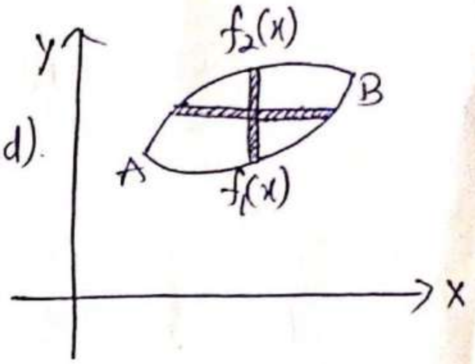
Subject Name: Mathematics – II

Topic Name: Area Enclosed by plane curves.

Name of the faculty: V. Sankar Rao   Unit–III Sem–I/II

Cartesian co-ordinates.

Introduction :- Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$, intersecting in the points $A(a,c)$ and $B(b,d)$.



Consider a strip parallel to the ~~Strip~~ y-axis. On this strip y-varies from $y = f_1(x)$ to $y = f_2(x)$. When this strip moves parallel to itself, x varies from $x = a$ to $x = b$.

In this way the whole are is obtained

$$\therefore \text{Area } \boxed{A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\, dx.}$$

→ If we consider a strip parallel to the x-axis, then on this strip x varies from $x = \phi_1(y)$ to $x = \phi_2(y)$.

When the strip moves parallel to itself,

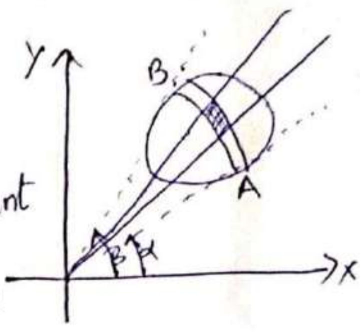
$\therefore$ y varies from $y = c$ to $y = d$.

In this way the whole area is obtained.

$$\text{Area } \boxed{A = \int_c^d \int_{\phi_1(y)}^{\phi_2(y)} dx\, dy}$$

Polar co-ordinates:

Consider the area enclosed by two curves $r=f_1(\theta)$ and $r=f_2(\theta)$ intersecting in the points $A(r_1,\alpha)$ and $B(r_2,\beta)$.



We divide the area into small areas by taking lines $\theta=$ constant and circles $r=$ constant. Then the area of the elementary rectangle (shaded area) is $r\,d\theta\,dr$.

$\therefore$ The required Area $\boxed{A=\int_{\alpha}^{\beta}\int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta.}$

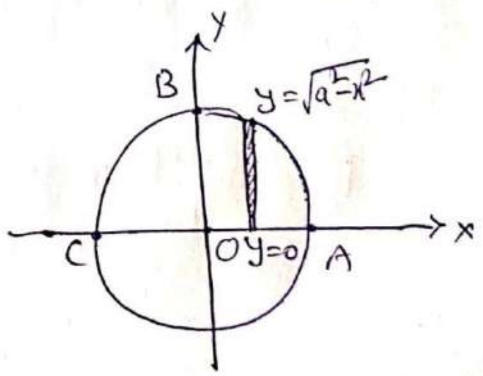
problems;-

imp (1) Find the area of the circle using integral.

Sol;- Let a be the radius of the circle.

Taking CA as the x-axis and its mid-point O as the orgin, the equation of the circle ABC is

$$x^2+y^2=a^2$$



To compute the area of OAB.

Consider the strip parallel to the y-axis

on this strip y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

Then the strip moves from $x=0$ to $x=a$.

$\therefore$ Required Area $=4$ (Area OAB).

$$\text{Area} = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} dy\,dx$$

$$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\cdot\int_0^a \left(\sqrt{a^2-x^2}\right) dx$$

$$= 4\cdot\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4\cdot\left[0+\frac{a^2}{2}\sin^{-1}(1)-(0+0)\right]$$

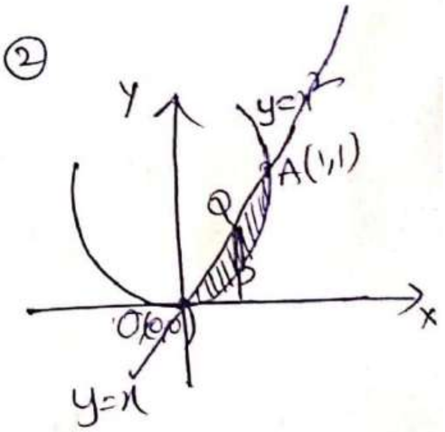
$$= \cancel{4}^{2}\,\frac{a^2}{\cancel{2}}\cdot\frac{\pi}{\cancel{2}} = \pi a^2$$

$$\therefore \boxed{\text{Area} = \pi a^2}.$$

2). Find the area bounded by the curves, $y=x, y=x^2$.

Sol:- Given Curves are $y = x \longrightarrow ①$

$y = x^2 \longrightarrow ②$



To find their points of intersection,

Solve eq ① and eq ② we get

$x^2 = x \Rightarrow (x^2-x)=0$

$x(x-1)=0$

$x=0,\ x=1.$

when $x=0,\ y=0$

$x=1,\ y=1.$

Thus the given curves intersect at the points $O(0,0), A(1,1)$.

$\therefore$ Area bounded by the curves, y varies from $y=\sqrt{x}$ to $y=x$

x varies from $x=0$ to $x=1$

Hence the required Area $= \int_{x=0}^{1}\int_{y=\sqrt{x}}^{x} dy\,dx$

$$= \int_0^1 [y]_{\sqrt{x}}^{x}\,dx$$

$$= \int_0^1 (x-\sqrt{x})dx = \int_0^1 (x - x^{1/2})\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^{3/2}}{3/2}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3}(1)\right) - 0$$

$$= \left|\frac{1}{2} - \frac{2}{3}\right|$$

$$= \left|\frac{3-4}{6}\right| = \frac{1}{6}.$$

3). Find the double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Sol:- given curves are $y = 4x - x^2$ —— ①

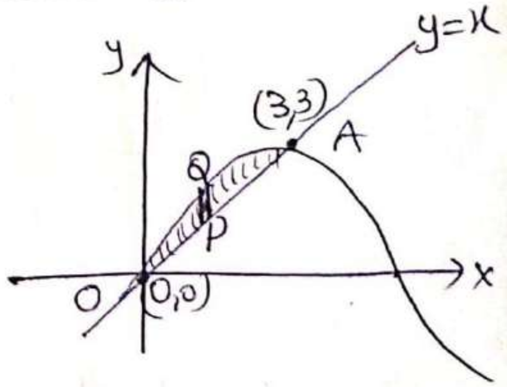
$y = x$ —— ②

The parabola $y = 4x - x^2$ can be written as

$$y - 4 = -(x^2 - 4x + 4) = -(x-2)^2$$

$$\therefore (x-2)^2 = -(y-4).$$

So the vertex is $(2,4)$ and it opens downwards.



Substituting eq ② in eq ① we get

$$x = 4x - x^2$$

$$\Rightarrow 3x - x^2 = 0$$

$$x(3-x) = 0$$

$$\Rightarrow x = 0,\ x = 3.$$

when $x=0, \quad y=0$

$x=3, \quad y=3.$

$\therefore$ The points of intersection are $(0,0)$ and $(3,3)$.

Consider a strip PQ parallel to the y-axis.

on this strip y varies from $y=x$ to $y=4x-x^2$

x varies from $x=0$ to $x=3$.

Hence the required Area $= \int_{x=0}^{3}\int_{y=x}^{4x-x^2} dy\,dx$

$$= \int_0^3 [y]_x^{4x-x^2}\,dx$$

$$= \int_0^3 \left((4x-x^2)-x\right)dx$$

$$= \int_0^3 (3x-x^2)\,dx$$

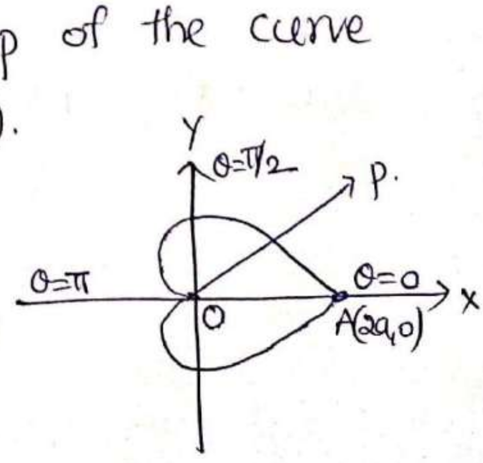
$$= 3\left(\frac{x^2}{2}\right)_0^3 - \left(\frac{x^3}{3}\right)_0^3$$

$$= 3\left(\frac{9}{2}\right) - \frac{27}{3}$$

$$= \frac{27}{2} - 9 \Rightarrow \frac{27-18}{2} = \frac{9}{2} \text{ sq.units.}$$

4). Find the area of the loop of the curve

$r = a(1+\cos\theta)$.



Sol:- The cardioid $r=a(1+\cos\theta)$ is symmetrical about the intial line and passes through the pole O.

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It intersects the initial line at the points O and A(2a, 0).

As θ varies from $\theta = 0$ to $\theta = \pi$, r increases from 0 to 2a.

To determine the polar limits of integration for the Area A, imagine radius vector through A from O which emerges at the point P. where $r = a(1+\cos\theta)$, such radii vectors are drawn between the limits $\theta = -\pi$ to $\theta = 0$.

Hence the required Area = 2 x Area of the curve above the initial line.

$$= 2 \iint_A r\,dr\,d\theta$$

$$= 2 \cdot \int_{\theta=-\pi}^{0} \int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2 \cdot \int_{-\pi}^{0} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \cancel{2} \cdot \frac{1}{\cancel{2}} \int_{-\pi}^{0} (a(1+\cos\theta))^2 d\theta$$

$$= a^2 \int_{-\pi}^{0} (1+\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= a^2 \int_{-\pi}^{0} \left(1 + \frac{1+\cos 2\theta}{2} + 2\cos\theta\right) d\theta$$

$$= a^2 \int_{-\pi}^{0} \left(\frac{3}{2} + \frac{1}{2}\cos 2\theta + 2\cos\theta\right) d\theta$$

$$= a^2 \left[\frac{3}{2}(\theta) + \frac{1}{2} \cdot \frac{\sin 2\theta}{2} + 2 \cdot \sin\theta\right]_{-\pi}^{0}$$

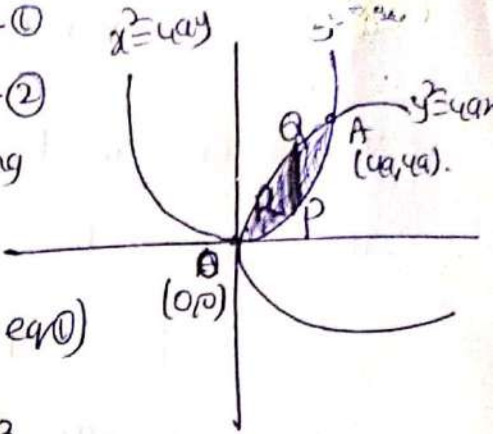
$$= a^2 \left[\cancel{\frac{3}{2}} 0 - \left(\frac{3}{2}(-\pi) + \frac{1}{2} \frac{\sin 2(-\pi)}{2} + 2(\sin(-\pi)\right)\right.$$

$$= a^2\left(\frac{3}{2}\pi + 0 + 0\right) = \frac{3\pi a^2}{2}$$

5). Find the area of the region bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol:- given curves are $y^2 = 4ax$ ——— ①

$x^2 = 4ay$ ——— ②



To find their points of intersections solving eq ① and eq ②

Squaring, eq ② $x^4 = 16a^2y^2$ (Using eq①)

$x^4 = 16a^2(4ax)$

$x^4 = 64a^3x \Rightarrow x^4 - 64a^3x = 0$

$x(x^3 - 64a^3) = 0$

$x = 0, \; x^3 = 64a^3$

$x^3 = (4a)^3$

$x = 4a.$

∴ when $x = 0, \; y = 0$

$x = 4a, \; y = 4a.$ ∴ The points of intersection are $O(0,0)$ and $A(4a,4a)$.

→ The region R can be covered by varying x from the upper curve $x = y^2/4a$ to the lower curve while y varies from 0 to $4a$.

(or) Consider the strip parallel to y-axis. on this strip

y varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$.

The strip moves from $x = 0$ to $x = 4a$.

Hence the required Area $A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$

$$A = \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} \, dx$$

$$\text{Area} = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \int_0^{4a} \left(2\sqrt{a}\cdot x^{1/2} - \frac{1}{4a}x^2\right) dx$$

$$= 2\sqrt{a}\cdot\left(\frac{x^{3/2}}{3/2}\right)_0^{4a} - \frac{1}{4a}\left(\frac{x^3}{3}\right)_0^{4a}$$

$$= 2\cdot a^{1/2}\cdot\frac{2}{3}\cdot(4a)^{3/2} - \frac{1}{\cancel{4a}}\frac{(4a)^{3/2}}{3}$$

$$= \frac{4}{3}\cdot a^{1/2}\cdot a^{3/2}\cdot 4^{3/2} - \frac{16a^2}{3}$$

$$= \frac{4}{3}\cdot a^{1/2+3/2}\cdot 4(2)^{\cancel{2}\cdot 3/\cancel{2}} - \frac{16a^2}{3}$$

$$= 8\cdot\frac{4}{3}a^2 - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\boxed{\text{Required Area} = \frac{16a^2}{3}}.$$

Practice Questions :-

1). Find the area enclosed by the parabolas $x^2 = y$ and $y^2 = x$.

2). Find by double integration the area bounded by the curves $y = 2-x$ and $y^2 = 2(2-x)$.

3). Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$

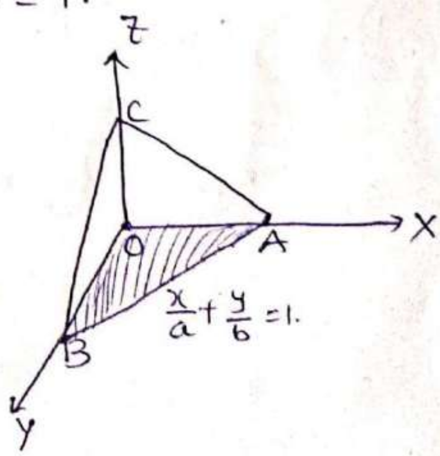
—·x—

Introduction:- Let $z = f(x,y)$ be a surface above the xy-plane. The volume V of the solid bounded by the surface $z = f(x,y)$ where orthogonal projection on the xy-plane is a closed curve enclosing a surface S is given by $\boxed{V = \iint_S z\,dx\,dy}$ (or) $\boxed{\iint_S f(x,y)\,dx\,dy}$.

Note:- If the base area is represented through polar co-ordinates, the required volume will be

$$\iint f(x,y)\, r\, dr\, d\theta \text{ where } x = r\cos\theta,\ y = r\sin\theta.$$

Problems:- (1). Using double integration, find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.



Sol:- Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ —— ①

ie $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$ —— ②

in the xy-plane, $z = 0$.

Substituting $z = 0$ in eq ①

we get $\frac{x}{a} + \frac{y}{b} = 1$.

→ The Region R (Shaded portion) in the xy-plane is a triangle OAB bounded by $x=0$, $y=0$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

In this region x varies from $x=0$ to $x=a$.

y varies from $y=0$ to $y=b\left(1-\frac{x}{a}\right)$.

Hence the required volume $= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} z \, dy \, dx.$

$$= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx. \quad \text{(from eq②)}$$

$$= c\int_{0}^{a}\left[y-\frac{x}{a}(y)-\frac{1}{b}\left(\frac{y^2}{2}\right)\right]_{0}^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_{0}^{a} b\left(1-\frac{x}{a}\right)-\frac{x}{a}\left(b\left(1-\frac{x}{a}\right)\right)-\frac{1}{2b}\left(b\left(1-\frac{x}{a}\right)\right)^2 dx$$

$$= c\int_{0}^{a} b-\frac{bx}{a}-\frac{bx}{a}+\frac{bx^2}{a^2}-\frac{1}{2\cancel{b}}\,b^{\cancel{2}}\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right) d$$

$$= c\int_{0}^{a}\left(b-\frac{2bx}{a}+\frac{bx^2}{a^2}-\frac{b}{2}-\frac{bx^2}{2a^2}+\frac{bx}{a}\right)dx$$

$$= c\int_{0}^{a}\left(\frac{b}{2}-\frac{bx}{a}+\frac{1}{2}\frac{bx^2}{a^2}\right)dx$$

$$= c\left[\frac{b}{2}(x)-\frac{b}{a}\left(\frac{x^2}{2}\right)+\frac{b}{2a^2}\left(\frac{x^3}{3}\right)\right]_{0}^{a}$$

$$= c\left[\frac{b}{2}(a)-\frac{b}{\cancel{a}}\left(\frac{a^{\cancel{2}}}{2}\right)+\frac{b}{2\cancel{a^2}}\left(\frac{a^{\cancel{3}}}{3}\right)\right]$$

$$= \frac{ab\cancel{c}}{2}-\frac{a\cancel{bc}}{2}+\frac{abc}{6} = \frac{abc}{6}.$$

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2). Find the volume bounded by the cylinders $x^2+y^2=4$, $y+z=4$ and $z=0$.

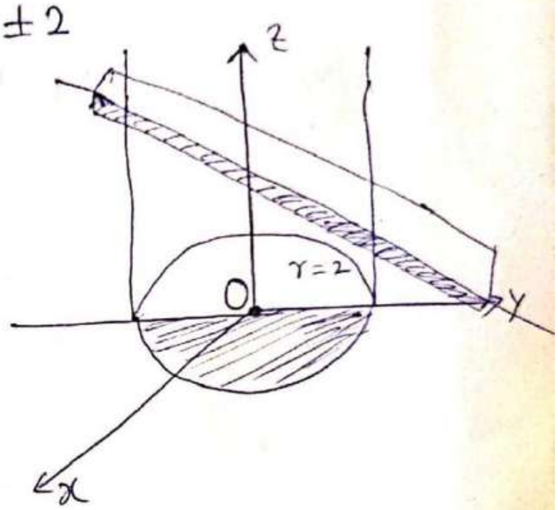
Sol:- Equation of the cylinder is $x^2+y^2=4$ and $z=0$.

$\therefore \quad x = \pm\sqrt{4-y^2}$.

now $x=0, \Rightarrow 4-y^2=0$

$y^2=4 \Rightarrow y=\sqrt{4}$.

$y=\pm 2$



The base is a circle in xy-plane with centre at the origin and radius 2 units.

Equation of plane in the space is given by $y+z=4 \Rightarrow \boxed{z=4-y.}$

The limits of x are given by $x=\pm\sqrt{4-y^2}$

ie $x=-\sqrt{4-y^2}$ to $x=\sqrt{4-y^2}$

and $x=0 \Rightarrow y=\pm 2$.

Required volume $= \iint_S z \, dx\, dy$

$$= \int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4-y)\, dy\, dx.$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\, dx\, dy.$$

$$= 2.\int_{-2}^{2} (4-y)\,[x]_0^{\sqrt{4-y^2}}\, dy.$$

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$$\text{Required volume} = 2\cdot\int_{-2}^{2}(4-y)\cdot\sqrt{4-y^2}\,dy.$$

$$= 2\left[\int_{-2}^{2} 4\cdot\sqrt{4-y^2}\,dy - \int_{-2}^{2} y\sqrt{4-y^2}\,dy\right]$$

(Second integral is zero as the integrand is an odd function).

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy. \qquad \left[\text{Integrand is even function}\right].$$

$$= 8(2)\int_{0}^{2}\sqrt{2^2-y^2}\,dy.$$

$$= 16\cdot\left[\frac{y}{2}\cdot\sqrt{4-y^2} + \frac{\overset{2}{\cancel{4}}}{\cancel{2}}\sin^{-1}\left(\frac{y}{2}\right)\right]_0^2$$

(using $\int\sqrt{a^2-x^2}\,dx$ formula)

$$= 16\left[\left(0+2\sin^{-1}(1)\right)-(0+0)\right]$$

$$= 16\cdot\cancel{2}\left(\pi/\cancel{2}\right)$$

$$\boxed{\text{Required volume} = 16\pi.}$$

Practice Questions:

1) Find by double integration the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

2). Find the volume bounded by xy-plane, the cylinder $x^2+y^2=1$ and the plane $2x+3y+4z=12$.

Subject code : MA201BS Lecture no: 11 (1) Date: 2(6/21

Subject Name: Mathematics - II.

Topic Name: Volume as a Triple Integral.

Name of the faculty: V. Sankar Rao UNIT-III Sem-I/II.

Introduction :- Suppose a three dimensional solid is cut into elemental rectangular paralleopiped by drawing planes parallel to the co-ordinate planes. The volume of an elemental paralleopiped δV is $\delta x\, \delta y\, \delta z$.

Hence the total volume of solid is $\iiint_V dv = \iiint_V dx\,dy\,dz$

where the integration is carried over the entire volume.

Problems :-

(1) Using triple integral, find the volume of the sphere whose radius is 'a' units.

Sol :- The equation of the sphere is $x^2+y^2+z^2=a^2$

In cartesian form the region of integration is as follows.

z varies from $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$

y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

x varies from $-a$ to a.

Required volume $= \iiint_V dx\,dy\,dz$

$$= \int_{-a}^{a}\int_{-\sqrt{a-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{+\sqrt{a^2-x^2-y^2}} dx\,dy\,dz.$$

$$\text{Required volume} = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dx\, dy\, dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\, dx\, dy.$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2-y^2}+\sqrt{a^2-x^2-y^2}\right) dx\, dy.$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\, dx\, dy.$$

$$= 2\int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2-(y)^2}\, dy\right] dx$$

(Since integrand is even function)

$$= (2)(2)\int_{-a}^{a}\left[\int_{0}^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2-(y)^2}\, dy\right] dx$$

$$= 4.\int_{-a}^{a}\left[\frac{y}{2}\sqrt{(a^2-x^2)-y^2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 4\int_{-a}^{a}\left(\left(0+\frac{a^2-x^2}{2}\sin^{-1}(1)\right)-0\right) dx$$

$$= 4.\int_{-a}^{a}\frac{(a^2-x^2)}{2}.\frac{\pi}{2}\, dx \quad \text{(Integrand is even function)}$$

$$= \frac{4\pi}{4}.2\int_{0}^{a}(a^2-x^2)\,dx$$

$$= 2\pi\left[a^2(x)-\frac{x^3}{3}\right]_0^a = 2\pi\left[(a)a^2-\frac{a^3}{3}\right] = 2\pi\left(a^3-\frac{a^3}{3}\right)$$

$$= 2\pi\left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi a^3$$

2). Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Sol :- given cylinders are $x^2+y^2=a^2 \longrightarrow (1)$ and

$x^2+z^2=a^2 \longrightarrow (2)$

from eq (1) $y^2 = a^2-x^2$

$y = \pm\sqrt{a^2-x^2}$ and

from eq (2) $z^2 = a^2-x^2$

$z = \pm\sqrt{a^2-x^2}$

Required volume can be covered as follows.

z : from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

y : from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

x : from $-a$ to a.

Thus the volume V enclosed by the cylinders

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\, dy\, dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx.$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}-\left(-\sqrt{a^2-x^2}\right)\right) dy\, dx$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}\right) dy\, dx.$$

$$= 2\int_{-a}^{a} \left(\sqrt{a^2-x^2}\right)[y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx.$$

(3)

Required volume $= 2\int_{-a}^{a} \sqrt{a^2-x^2}\left(\sqrt{a^2-x^2}-\left(-\sqrt{a^2-x^2}\right)\right)dx$

$= (2)(2)\int_{-a}^{a}\left(\sqrt{a^2-x^2}\right)\left(\sqrt{a^2-x^2}\right)dx$

$= 4.\int_{-a}^{a}(a^2-x^2)\,dx$ (Integrand is even function)

$= 4.(2)\int_{0}^{a}(a^2-x^2)\,dx$

$= 8.\left[a^2(x)-\frac{x^3}{3}\right]_0^a$

$= 8.\left[a^2(a)-\frac{a^3}{3}\right].$

$= 8\left(a^3-\frac{a^3}{3}\right)$

$= 8a^3\left(1-\frac{1}{3}\right)$

$= 8a^3\frac{(2)}{3}$

$\boxed{\text{Required volume} = \frac{16a^3}{3}}$.

imp

3). Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

(OR)

Find the volume of the greatest rectangular paralleopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Sol:- given ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is cut into 8 equal parts by the three co-ordinate planes.

Hence the volume of the solid is equal to 8 times the volume of the solid bounded by $x=0, y=0, z=0$ to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

$\because$ For a fixed a point (x,y) on the xy-plane,

z: from $z=0$ to $z = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

consider the quardnt of the ellipse in the first quadrant of the xy-plane. For a fixed x, y: $y=0$ to $b\sqrt{1-\frac{x^2}{a^2}}$.

Then x: from 0 to a.

$$\therefore \text{Required volume} = 8 \cdot \int_{x=0}^{a}\int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz$$

$$= 8 \cdot \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\, dy\,dx$$

$$= 8 \cdot \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\,dx.$$

$\left[\text{write } 1-\frac{x^2}{a^2} = \frac{p^2}{b^2}\right]$

$$= 8c \int_0^a \left[\int_0^p \frac{1}{b}\sqrt{p^2-y^2}\,dy\right] dx.$$

put $y = p\sin\theta$

$dy = p\cos\theta\, d\theta$

If $y=0$, $\theta = 0$.

$y = p$, $\theta = \pi/2$.

$$= \frac{8c}{b} \cdot \int_0^a \left[\int_0^{\pi/2} p\cdot\cos\theta \cdot p\cos\theta\, d\theta\right] dx$$

$$= \frac{8c}{b} \left[\int_0^a\int_0^{\pi/2} p^2\cos^2\theta\, d\theta\right] dx.$$

$$= \frac{8c}{b} \int_0^a p^2\left[\frac{\pi}{4}\right] dx \qquad \left(\because p^2 = b^2\left(1-\frac{x^2}{a^2}\right)\right)$$

$$= \frac{\cancel{8}^{2}c}{b} \cdot \frac{\pi}{\cancel{4}} \int_0^a b^2\left(1-\frac{x^2}{a^2}\right) dx$$

$$= \frac{2c\pi}{\cancel{b}} \cdot b^{\cancel{2}} \int_0^a \left(1-\frac{x^2}{a^2}\right) dx = 2c\pi(b)\left[x - \frac{1}{a^2}\cdot\frac{x^3}{3}\right]_0^a$$

$$= 2\pi bc \cdot \left[a - \frac{1}{a^2}\cdot\frac{a^3}{3}\right]$$

$$= 2\pi bc\left[\frac{2a}{3}\right] = \frac{4\pi abc}{3}.$$

Subject code : MA201BS Lecture No: 12 (1) Date: 28/6/21

Subject Name : Mathematics - II.

Topic Name : Change of Variables in Triple integrals.

Name of the faculty: V. Sankar Rao Unit - III Sem - I/II.

Introduction:-

(I) Changing from cartesian to cylindrical polar co-ordinates.

Let P be a point in space whose cartesian co-ordinates are (x, y, z).

Let (r, θ, z) be the cylindrical polar co-ordinates of P.

Then $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

and $dx\,dy\,dz$ is to be replaced by

$$\boxed{|J|\, dr\, d\theta\, dz = r\, dr\, d\theta\, dz}$$

where $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

$|J| = r.$

Note:- Cylindrical polar coordinates are useful if the region of integration is a right circular cylinder.

(II). Changing from Cartesian to spherical polar co-ordinates:

Let P be a point in a space whose cartesian co-ordinates are (x, y, z) and whose spherical co-ordinates are (r, θ, ϕ).

Then $x = r\cos\theta\cos\phi$, $y = r\sin\theta . \sin\phi$, and $z = r\cos\theta$.

$\therefore$ $dx\,dy\,dz$ is to be replaced by $\boxed{|J|\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi}$.

Where
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$J = \begin{vmatrix} \sin\theta . \cos\phi & r\cos\theta . \cos\phi & -r\sin\theta . \sin\phi \\ \sin\theta . \sin\phi & r\cos\theta . \sin\phi & r\sin\theta . \cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\boxed{|J| = r^2 . \sin\theta}$$

<u>Note</u>: Spherical polar co-ordinates are useful when the region of integration is a part of the sphere.

Problems :-

(1). Evaluate $\iiint (x^2+y^2+z^2)\, dx\, dy\, dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transforming into spherical polar co-ordinates.

Sol:- Converting the given integral into spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ we have

$$x^2+y^2+z^2 = r^2, \quad J = r^2\sin\theta.$$

Thus $$\iiint (x^2+y^2+z^2)\, dx\, dy\, dz = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^2(r^2\sin\theta\, dr\, d\theta\, d\phi)$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\frac{r^5}{5}\right)_0^1 \sin\theta\, d\theta\, d\phi.$$

$$= \frac{1}{5}\int_0^{2\pi}(-\cos\theta)_0^{\pi}\, d\phi$$

$$= -\frac{1}{5}\int_0^{2\pi}(\cos\pi - 1)\, d\phi$$

$$= -\frac{1}{5}(-2)\,[\phi]_0^{2\pi}$$

$$= \frac{2}{5}(2\pi - 0) \Rightarrow \frac{4\pi}{5}.$$

imp

2). Using cylindrical co-ordinates, find the volume of the sphere $x^2+y^2+z^2=a^2$.

Sol:- Let the equation of the sphere be $x^2+y^2+z^2=a^2$ —— ①

Now, we will solve this problem using cylindrical co-ordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

So, $x^2+y^2=r^2(\cos^2\theta+\sin^2\theta)=r^2$

and $x^2+y^2+z^2=a^2 \Rightarrow r^2+z^2=a^2$

Limits of z are : $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$ ie $-\sqrt{a^2-r^2}$ to $\sqrt{a^2-r^2}$

Limits of r are : 0 to a. and

limits of θ are : 0 to 2π.

Required volume : $\iiint dx\,dy\,dz = \iiint (r\,d\theta\,dr)\,dz$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,d\theta\,dr\,dz$$

$$= 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} dz\; r\,d\theta\,dr$$

$$= 2\cdot\int_0^{2\pi}\int_0^{a}[z]_0^{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$= 2\cdot\int_0^{2\pi}\int_0^{a}\left(r\cdot\sqrt{a^2-r^2}\right)dr\,d\theta.$$

$$= (-1)\int_0^{2\pi}\left[\int_0^{a}(a^2-r^2)^{1/2}(-2r)\,dr\right]d\theta$$

$$= -\int_0^{2\pi}\left[(a^2-r^2)^{3/2}\left(\frac{2}{3}\right)\right]_0^{a} d\theta$$

$$= -\frac{2}{3}\left[\int_0^{2\pi}(0-a^3)\,d\theta\right] \qquad \left[\int f(x)^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}\right]$$

$$= 2/3\int_0^{2\pi} a^3\,d\theta = \frac{2}{3}a^3[\theta]_0^{2\pi} = \frac{2}{3}[2\pi]a^3$$

$$= \frac{4}{3}\pi a^3.$$

3). Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dx\,dy\,dz$ by changing to spherical polar co-ordinates. (3)

Sol :- Changing to spherical polar co-ordinates by putting $x = r\sin\theta \cdot \cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ we have

$J = r^2\sin\theta$, $x^2+y^2+z^2 = r^2$ and $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$.

Also the given region of integration is the volume of the sphere $x^2+y^2+z^2=1$ in the positive octant for which r varies from 0 to 1, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} \, dx\,dy\,dz$$

$$= \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \frac{r^2\sin\theta}{\sqrt{1-r^2}} \, dr\, d\theta\, d\phi.$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} \, dr \right] \sin\theta\, d\theta\, d\phi.$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \left\{ \frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right\} dr \sin\theta\, d\theta\, d\phi.$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left\{ \sin^{-1}(r) - \left\{ \frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1}(r) \right\} \right\}_0^1 \sin\theta.\, d\theta\, d\phi.$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2} \right) \sin\theta .\, d\phi\, d\theta.$$

$$= \frac{\pi}{4} \cdot \int_0^{\pi/2} \left[\int_0^{\pi/2} \sin\theta \, d\theta \right] d\phi$$

$$= \frac{\pi}{4} \cdot \int_0^{\pi/2} \left(-\cos\theta\right)_0^{\pi/2} d\phi.$$

$$= \frac{\pi}{4} \int_0^{\pi/2} d\phi$$

$$= \frac{\pi}{4} \left(\theta\right)_0^{\pi/2} = \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{8}.$$

Practice Questions:

1). Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ax$.

2). Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transform into spherical polar co-ordinates.

Subject code : MA201BS Lecture No : 13 Date : 30/6/21

Subject Name : Mathematics-II.

Topic Name : Calucculation of Mass.

Name of the faculty : V. Sankar Rao Unit-III , Sem- I/II

Introduction :- If the surface density ρ of a plane lamina varies from point to point of the lamina and if it can be expressed as a function of the co-ordinates of a point then the mass of an elementary area dA is $\rho\, dA$.

Let $\rho(x,y)$ be the density at any point $p(x,y)$ of a plane lamina in the xy-plane. The elemental mass of elemental mass area around p is given by $dm = \rho(x,y)dxdy$ and the total mass of the lamina is given by $\boxed{\iint dm = \iint \rho(x,y)\,dxdy}$ integrated over the area of the lamina under consideration.

Mass of a Solid :- Let $\rho(x,y,z)$ be the density at any point $\rho(x,y,z)$ of a solid of mass. The elemental mass of an elemental volume around ρ is given by

$$\boxed{\iiint dm = \iiint \rho(x,y,z)\, dx\, dy\, dz}$$ integrated over the entire volume of the solid.

Note :- In polar co-ordinates, taking $\rho = \phi(r,\theta)$ at the point $p(r,\theta)$, the total mass of the lamina is given by

$$\boxed{\iint dm = \iint \rho(r,\theta)\, r\, d\theta\, dr}.$$

Problems:-

(1) A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by λxy, find by double integration, the mass of the lamina.

Sol:- The given curves are $y = x^2 - 3x \longrightarrow ①$

ie $y + \frac{9}{4} = x^2 - 2 \cdot \frac{3}{2} x + \frac{9}{4}$ (adding $\frac{9}{4}$ on both sides)

$y + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2$ which is parabola and $y = 2x$. ②

Solving eq ① and eq ② we get

$x^2 - 3x = 2x$

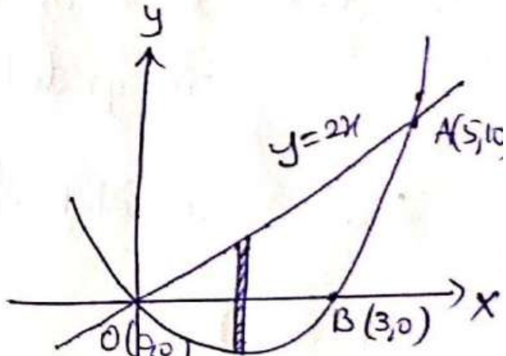
$x^2 - 5x = 0$

$x(x-5) = 0 \Rightarrow x = 0$ (or) $x = 5$

Also the curve $y = x^2 - 3x$ intersects the x-axis where

$y = 0 \Rightarrow x(x-3) = 0$

$x = 0, \; x = 3.$



$\therefore$ The given curves intersect at the points $O(0,0)$ and $A(5,10)$.

The lamina is the OAB. Taking vertical strip parallel to y-axis, the mass of the lamina

$$m = \iint dm = \iint \rho \, dx\,dy \qquad [\because \text{density}, \rho = \lambda xy].$$

$$= \int_{x=0}^{5} \int_{y=x^2-3x}^{2x} \lambda xy \, dx\,dy.$$

$$= \lambda \int_{0}^{5} x \cdot \left[\frac{y^2}{2}\right]_{x^2-3x}^{2x} dx.$$

$$m = \lambda \int_0^5 x\left[(2x)^2 - \frac{(x^2-3x)^2}{2}\right]dx$$

$$= \lambda \int_0^5 x\left(4x^2 - \frac{(x^4+9x^2-6x^3)}{2}\right)dx$$

$$= \lambda \int_0^5 x\left(\frac{8x^2-x^4-9x^2+6x^3}{2}\right)dx$$

$$= \frac{\lambda}{2}\int_0^5 (8x^3-x^5-9x^3+6x^4)\,dx$$

$$= \frac{\lambda}{2}\left[8\left(\frac{x^4}{4}\right)_0^5 - \left(\frac{x^6}{6}\right)_0^5 - 9\left(\frac{x^4}{4}\right)_0^5 + 6\left(\frac{x^5}{5}\right)_0^5\right]$$

$$= \frac{\lambda}{2}\left[8\left(\frac{5^4}{4}\right) - \left(\frac{5^6}{6}\right) - 9\left(\frac{5^4}{4}\right) + 6\left(\frac{5^5}{5}\right)\right]$$

$$= \frac{\lambda}{2}\left[5^4\left(\frac{8}{4}\right) - \frac{5^2}{6} - \frac{9}{4} + 6\left(\frac{5}{5}\right)\right]$$

$$= \frac{\lambda}{2}5^4\left(2 - \frac{25}{6} - \frac{9}{4} + 6\right)$$

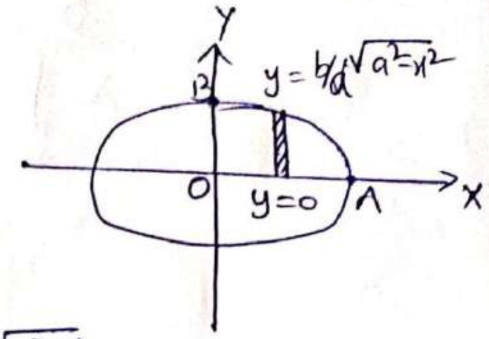
$$= \frac{\lambda}{2}5^4\left(\frac{8}{1} - \frac{25}{6} - \frac{9}{4}\right).$$

$$= \frac{\lambda}{2}(625)\left(\frac{192-100-54}{24}\right)$$

$$m. = \frac{\lambda}{2}(625)\left(\frac{38}{24}\right) = \frac{11,875\lambda}{24}.$$

2). Find the mass of the lamina in the form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose density at any point varies as the ~~point~~ product of the distances from the axes of the ellipse.

Sol:- Let the required mass be M which is four times the mass in the first quadrant.



from the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = b^2\left(\frac{a^2-x^2}{a^2}\right) \Rightarrow \pm \frac{b}{a}\sqrt{a^2-x^2}.$$

For the region OAB, x varies from 0 to a

y varies from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$.

Hence density $\rho = kxy$.

$$\therefore M = 4\int \rho\, dx\, dy = 4\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} kxy\, dx\, dy.$$

$$= 4k\int_0^a x\cdot\left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}} dx = \frac{\cancel{4}^{2}k}{\cancel{2}}\int_0^a x\cdot\left(\frac{b}{a}\left(\sqrt{a^2-x^2}\right)\right)^2 dx$$

$$= 2k\cdot\frac{b^2}{a^2}\int_0^a x\,(a^2-x^2)\,dx = \frac{2kb^2}{a^2}\int_0^a (xa^2-x^3)\,dx$$

$$= \frac{2kb^2}{a^2}\left[a^2\left(\frac{x^2}{2}\right)_0^a - \left(\frac{x^4}{4}\right)_0^a\right].$$

$$= 2k\frac{b^2}{a^2}\left(\frac{a^4}{2}-\frac{a^4}{4}\right) = \frac{\cancel{2}kb^2}{\cancel{a^2}}\left(\frac{a^{\cancel{4}\,2}}{\cancel{4}_2}\right) = \frac{a^2b^2k}{2}.$$

$$\therefore \boxed{M = \frac{ka^2b^2}{2}}.$$

Practice questions :-

imp 1). The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Find the mass the tetrahedron OABC if its denisity at any point is kyz.

imp 2). Find the mass of the lamina bounded by the curves $y^2=ax$ and $x^2=ay$ if the density of the lamina at any point varies as the square of its distance from the origin.

Subject Code : MA201BS. lecture no: 14 (1) Date: 2/7/21

Subject Name: Mathematics-II

Topic Name: Centre of Gravity.

Name of the faculty: V. Sankar Rao Unit-III Sem-I/II.

Introduction:

(a) To find the "Centre of Gravity" $(\bar{x},\bar{y})$ of a plane lamina, take the element of mass $\rho\,\delta x\,\delta y$ at the point $P(x,y)$.

Then
$$\bar{x} = \frac{\iint x\rho\,dx\,dy}{\iint \rho\,dx\,dy}, \quad \bar{y} = \frac{\iint y\rho\,dx\,dy}{\iint \rho\,dx\,dy}$$

with integrals embracing the whole lamina.

→ while using polar co-ordinates, take the elementary mass as $\rho r\,\delta\theta\,\delta r$ at the point $P(r,\theta)$ so that $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \bar{x} = \frac{\iint r\cos\theta\,\rho\,d\theta\,dr}{\iint \rho r\,d\theta\,dr}, \quad \bar{y} = \frac{\iint r\sin\theta\,\rho r\,d\theta\,dr}{\iint \rho r\,d\theta\,dr}.$$

(b) To find "centre of gravity" $(\bar{x},\bar{y},\bar{z})$ of a solid; take an element of mass $\rho\,\delta x.\delta y.\delta z$ enclosing the point $P(x,y,z)$. Then,

$$\bar{x} = \frac{\iiint x\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}, \quad \bar{y} = \frac{\iiint y\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz} \text{ and}$$

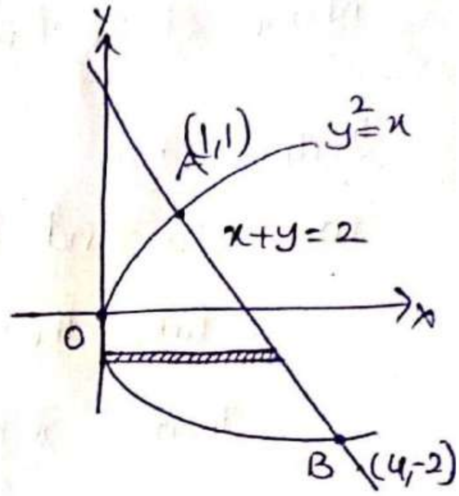
$$\bar{z} = \frac{\iiint z\rho\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}.$$

Problems :-

imp (1) Find the centre of gravity of the area bounded by the parabola $y^2=x$ and the line $x+y=2$.

Sol:- given $y^2=x \longrightarrow (1)$

$x+y=2 \longrightarrow (2)$



Solving eq (1) and eq (2), we get

Substituting eq (1) in eq (2), then

$y^2+y=2 \Rightarrow y^2+y-2=0.$

$y^2+2y-y-2=0$

$y(y+2)-1(y+2)=0$

$(y+2)(y-1)=0.$

$\therefore y=1,-2.$

when $y=1, \quad x=1.$

$y=-2, \quad x=4.$ $\therefore$ points of intersections are $A(1,1), \ B(4,-2)$

Let $G(\bar{x},\bar{y})$ be the centre of gravity of the lamina OAB,

So that $\bar{x}=\dfrac{\iint x\,\rho\,dx\,dy}{\iint \rho\,dx\,dy}, \quad \bar{y}=\dfrac{\iint y\,\rho\,dx\,dy}{\iint \rho\,dx\,dy}.$

when we have to find the centre of gravity of some area, we assume that the lamina has uniform density at any point of the lamina. $\therefore \boxed{\rho=1}$.

Thus $\bar{x}=\dfrac{\iint x\,dx\,dy}{\iint dx\,dy}$, and $\bar{y}=\dfrac{\iint y\,dx\,dy}{\iint dx\,dy}.$

Now, $\iint x \, dx\, dy = \int_{y=-2}^{1} \int_{x=y^2}^{2-y} x \, dx \, dy$

$$= \int_{-2}^{1} \left[\frac{x^2}{2}\right]_{y^2}^{2-y} dy$$

$$= \int_{-2}^{1} \left(\frac{(2-y)^2}{2} - \frac{y^4}{2}\right) dy.$$

$$= \frac{1}{2} \int_{-2}^{1} (2-y)^2 - y^4 \, dy.$$

$$= \frac{1}{2} \int_{-2}^{1} (4 - 4y + y^2 - y^4) \, dy.$$

$$= \frac{1}{2} \left(4y - 4 \cdot \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^5}{5}\right)_{-2}^{1}$$

$$\iint x \, dx \, dy = \frac{1}{2}\left[\left(\left(4 - \frac{4}{2} + \frac{1}{3} - \frac{1}{5}\right) - \left(4(-2) - 4\frac{(-2)^2}{2} + \frac{(-2)^3}{3} - \frac{(-2)^5}{5}\right]\right.\right.$$

$$= \frac{1}{2}\left[\frac{32}{15} + \frac{184}{15}\right] = \frac{36}{5}.$$

and $\iint y \, dx\, dy = \int_{y=-2}^{1} \int_{x=y^2}^{2-y} y \, dx\, dy.$

$$= \int_{-2}^{1} y \cdot [x]_{y^2}^{2-y} dy.$$

$$= \int_{-2}^{1} y(2 - y - y^2) \, dy.$$

$$= \int_{-2}^{1} (2y - y^2 - y^3) \, dy.$$

$$= \left(\cancel{2}\cdot\left(\frac{y^2}{\cancel{2}}\right) - \frac{y^3}{3} - \frac{y^4}{4}\right)_{-2}^{1}$$

$$= \left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left((-2)^2 - \frac{(-2)^3}{3} - \frac{(-2)^4}{4}\right).$$

$$\iint y\,dx\,dy = \left(\frac{5}{12}\right) - \left(4 + \frac{8}{3} - \frac{\cancel{16}^{4}}{\cancel{4}}\right) = \frac{5}{12} - \left(0 + \frac{8}{3}\right) = -\frac{9}{4}.$$

Also $\iint dx\,dy = \int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy$

$$= \int_{-2}^{1} [x]_{y^2}^{2-y}\,dy = \int_{-2}^{1} (2-y-y^2)\,dy.$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1}$$

$$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}\right).$$

$$= \frac{7}{6} - \left(-4 - \frac{\cancel{4}^{2}}{\cancel{2}} + \frac{8}{3}\right)$$

$$= \frac{7}{6} - \left(-6 + \frac{8}{3}\right)$$

$$\iint dx\,dy. = \frac{7}{6} + \frac{10}{3} = \frac{7+20}{6} = \frac{\cancel{27}^{9}}{\cancel{6}_{2}} = \frac{9}{2}.$$

$$\therefore\quad \bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} = \frac{\frac{36}{5}}{9/2} = \frac{\cancel{36}^{4}}{5} \times \frac{2}{\cancel{9}} = \frac{8}{5}.$$

$$\bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy} = \frac{-\frac{9}{4}}{9/2} = -\frac{\cancel{9}}{\cancel{4}_{2}} \times \frac{\cancel{2}}{\cancel{9}} = -1/2.$$

$\therefore$ Required centre of gravity $(\bar{x}, \bar{y}) = \left(\frac{8}{5}, \frac{-1}{2}\right)$

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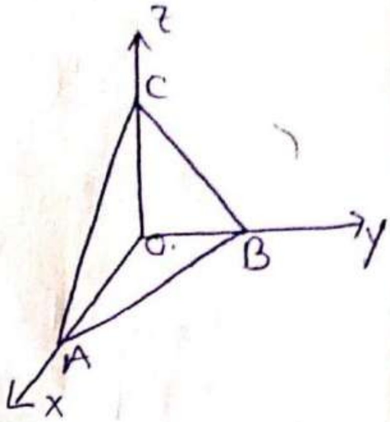
2) Find the mass, centroid of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol:- Let ρ be the constant density of the substance.



The elementary mass at $P = \rho\, dx\, dy\, dz$.

$\therefore$ The whole mass $M = \iiint \rho\, dx\, dy\, dz$.

the integrals embracing the whole volume OABC.

The limits for z: $z=0$ to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$.

The limits for y: $y=0$ to $y=b\left(1-\frac{x}{a}\right)$

The limits for x: $x=0$ to $x=a$.

$\therefore$ The required mass $M = \rho \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dx\, dy\, dz$.

$$M = \rho \int_0^a \int_0^{b(1-\frac{x}{a})} \left[z\right]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dx\, dy.$$

$$= \rho c \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right) dx\, dy.$$

$$= \rho c \int_0^a \left[\left(1-\frac{x}{a}\right)(y)-\frac{1}{b}\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})} dx.$$

$$= \rho c \int_0^a \left[\left(1-\frac{x}{a}\right) b\left(1-\frac{x}{a}\right)-\frac{1}{2\cancel{b}}\left(b\left(1-\frac{x}{a}\right)\right)^2\right] dx$$

$$= \rho bc \int_0^a \left(1-\frac{x}{a}\right)^2-\frac{1}{2}\left(1-\frac{x}{a}\right)^2 dx.$$

$$= \frac{\rho bc}{2} \int_0^a \left(1-\frac{x}{a}\right)^2 dx.$$

$$M = \frac{\rho bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{-\frac{3}{a}}\right]_0^a$$

$$= -\frac{\rho abc}{6}\left(\left(1-\frac{a}{a}\right)^3 - (1-0)^3\right)$$

$$= -\frac{\rho abc}{6}(0-1)$$

$$\boxed{M = \frac{\rho abc}{6}}.$$

∴ Let $G(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centroid. Then

$$\bar{x} = \frac{\iiint x \cdot \rho \, dx dy dz}{\iiint \rho \, dx dy dz},$$

$$\bar{y} = \frac{\iiint y \, \rho \, dx dy dz}{\iiint \rho \, dx dy dz},$$

$$\bar{z} = \frac{\iiint z \, \rho \, dx dy dz}{\iiint \rho \, dx dy dz}.$$

Here $M = \iiint \rho \, dx dy dz = \frac{\rho abc}{6}$.

$$\bar{x} = \rho \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x \, dz \, dy \, dx.$$

$$= \rho \int_0^a \int_0^{b(1-x/a)} x\,[z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy \, dx$$

$$= \rho \int_0^a \int_0^{b(1-x/a)} x \cdot c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy \, dx$$

$$= \rho c \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} \left(x - \frac{x^2}{a} - \frac{y}{b} \cdot x\right) dy \, dx.$$

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$$\bar{x} = ec \int_0^a \left(\left(x - \frac{x^2}{a}\right) y - \frac{x}{b}\frac{y^2}{2} \right)_0^{b(1-\frac{x}{a})} dx$$

$$= ec \int_0^a \left(x - \frac{x^2}{a}\right) b\left(1-\frac{x}{a}\right) - \frac{x}{2b}\left(b\left(1-\frac{x}{a}\right)\right)^2 dx$$

$$= ec \int_0^a bx\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right) - \frac{x}{2\cancel{b}} b^{\cancel{2}}\left(1-\frac{x}{a}\right)^2 dx$$

$$= e\,bc. \int_0^a x\left(1-\frac{x}{a}\right)^2 - \frac{1}{2} x\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{e\,bc}{2} \int_0^a x\left(1-\frac{x}{a}\right)^2 dx.$$

$$= \frac{e\,bc}{2} \int_0^a x\left(1+\frac{x^2}{a^2} - \frac{2x}{a}\right) dx$$

$$= \frac{e\,bc}{2} \int_0^a \left(x + \frac{1}{a^2}x^3 - \frac{2}{a}x^2\right) dx = \frac{e\,bc}{2}\left[\frac{x^2}{2} + \frac{1}{a^2}\frac{x^4}{4} - \frac{2}{a}\frac{x^3}{3}\right]_0^a$$

$$= \frac{e\,bc}{2}\left[\frac{a^2}{2} + \frac{1}{\cancel{a^2}}\frac{a^{\cancel{4}\,2}}{4} - \frac{2}{\cancel{a}}\cdot\frac{a^3}{3}\right] = \frac{ebc}{2}\left(\frac{a^2}{2} + \frac{a^2}{4} - \frac{2a^2}{3}\right).$$

$$\therefore\ \bar{x} = \frac{ea^2bc}{2}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right) = \frac{ea^2bc}{24}.\ ,\ \bar{x} = \frac{\frac{ea^2bc}{24}}{\frac{eabc}{6}}$$

Similarly, we can find $\boxed{\bar{y} = \frac{b}{4}}$, $\boxed{\bar{z} = \frac{c}{4}}$. $\therefore \boxed{\bar{x} = \frac{a}{4}}$.

$\therefore$ centroid $G = \left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$.

Practice Questions:

1). Find the centre of gravity of the region in the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2). Find the centre of gravity of the region in the first quadrant by the curve $x^{2/3} + y^{2/3} = a^{2/3}$

——— x ———

Summary: In this chapter we shall learn the method of daible integration to caluculate the area and Triple integrations, Applications of multiple integrals, that are

- Evaluation of double integrals (Cartesian & Polar coordinates)
- Change of order of Integration.
- change of variables for daeble integration.
- Evaluation of Triple Integrals.
- Change of variable for triple integration.
- Areas by double integration and volumes of double & Triple integration.
- Centre of mass and Gravity by double and triple integrals

Assignment Questions:

1). Find $\iint_R (x+y)^2\,dx\,dy$ over R bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

2). Change the order of integration and solve $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2\,dy\,dx$.

3). Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$.

4). Evaluate $\iiint_V z^2\,dx\,dy\,dz$ taken over the volume bounded by by the Surfaces $x^2+y^2=a^2$, $x^2+y^2=z$ and $z=0$.

5). The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Find the Mass of the tetrahedron OABC if its density at any point is $kxyz$.

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